Random Matrix Theory, Atmospheric Turbulence, and Free-Space Optical Communications

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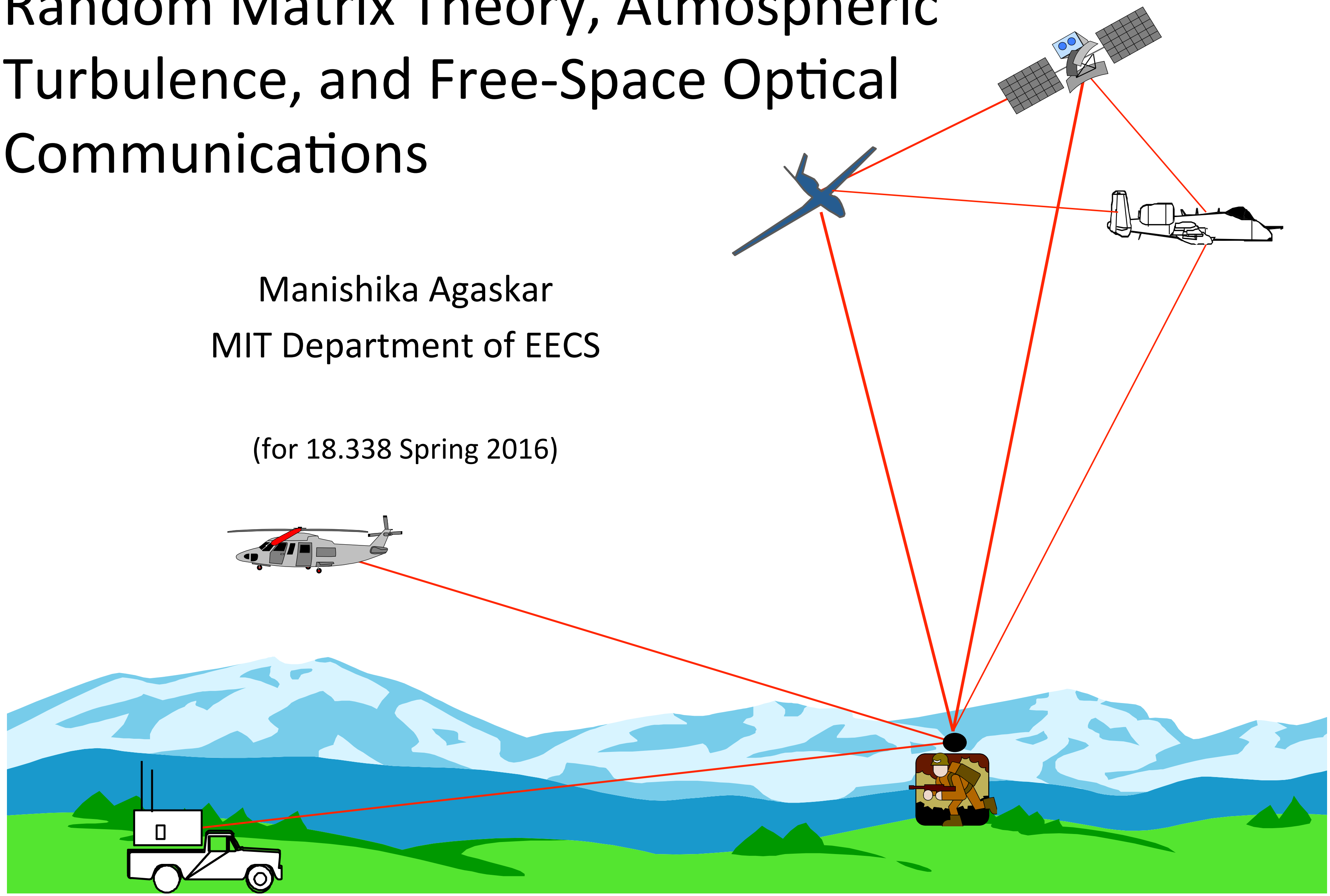


Image source: V.W.S. Chan, MIT 6.972

OUTLINE

1. Demonstrate the applicability of random matrix theory to free-space optical communications.
2. Use simulations to find out what assumptions are required to converge to RMT results with reasonably sized (i.e. not infinite) systems.
3. Use random matrix theory to find the lower limit on the achievable bit error rate in the presence of atmospheric turbulence.

Atmospheric Turbulence

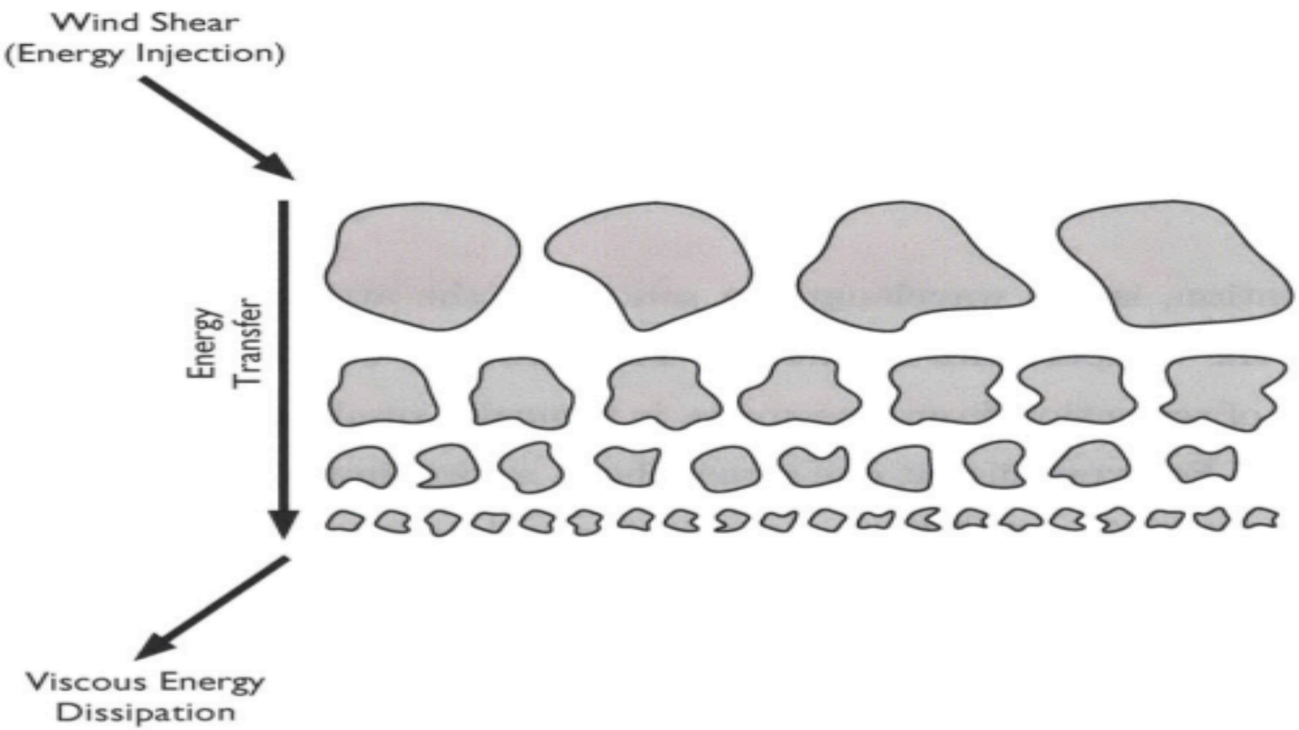
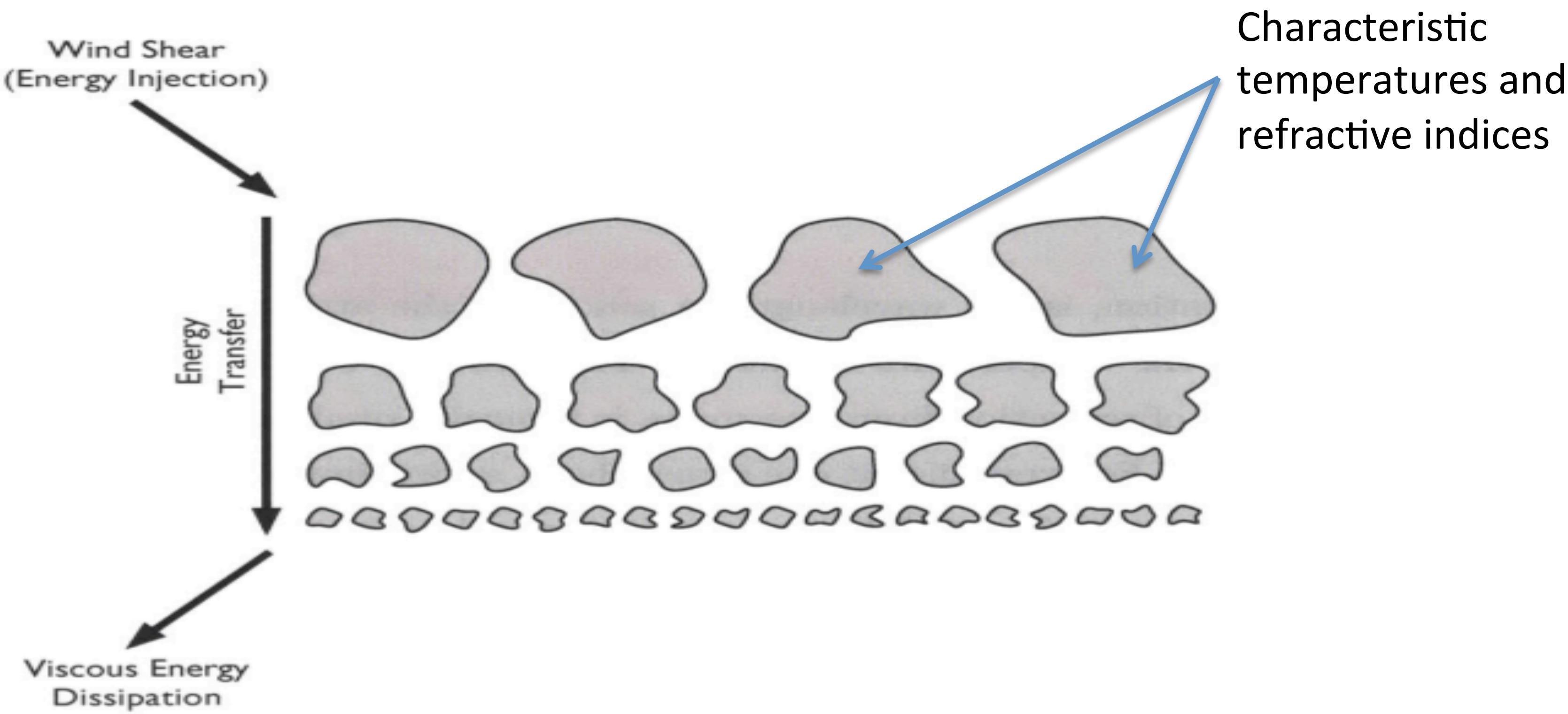


Image source: Puryear, PhD Thesis, 2011

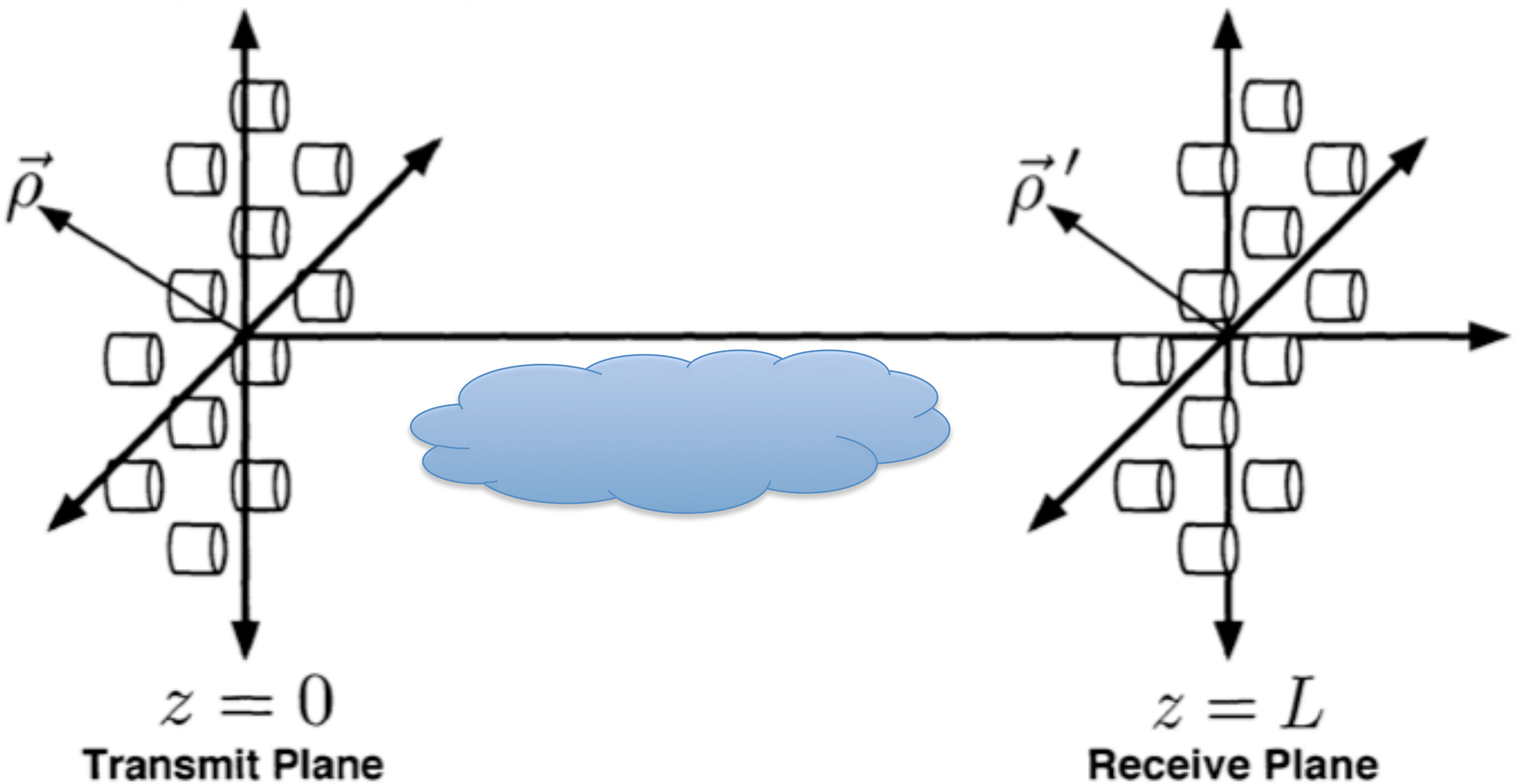
Atmospheric Turbulence



Atmospheric Turbulence

- Significant for optical communication because of small wavelength of laser light (≈1.5 μm)
  - → Changes the phase front of the beam
  - → Can cause deep fades in received power due to destructive interference at receiver
- Random fluctuations of index of refraction described by Kolmogorov model

Spatial Diversity via Sparse Apertures



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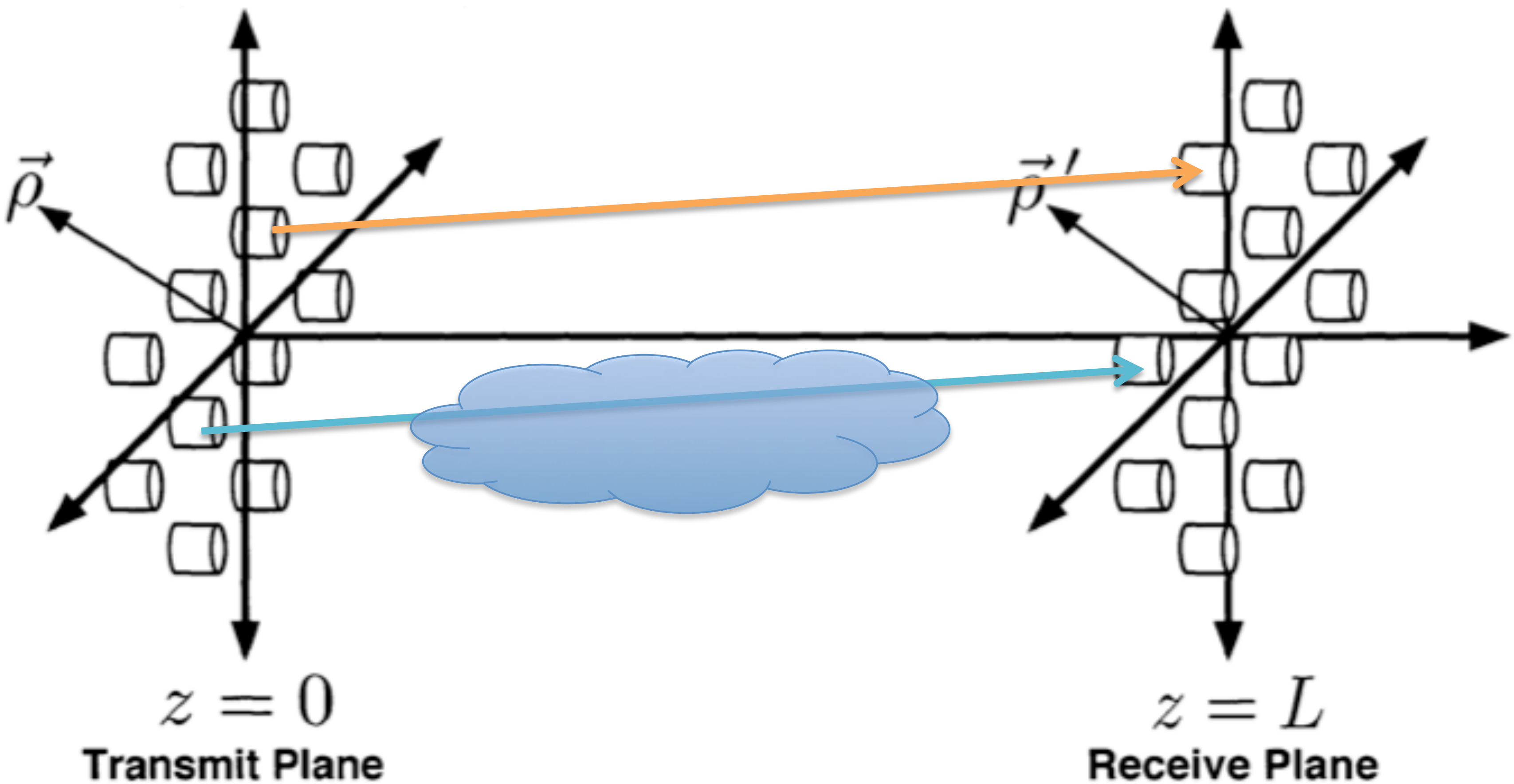
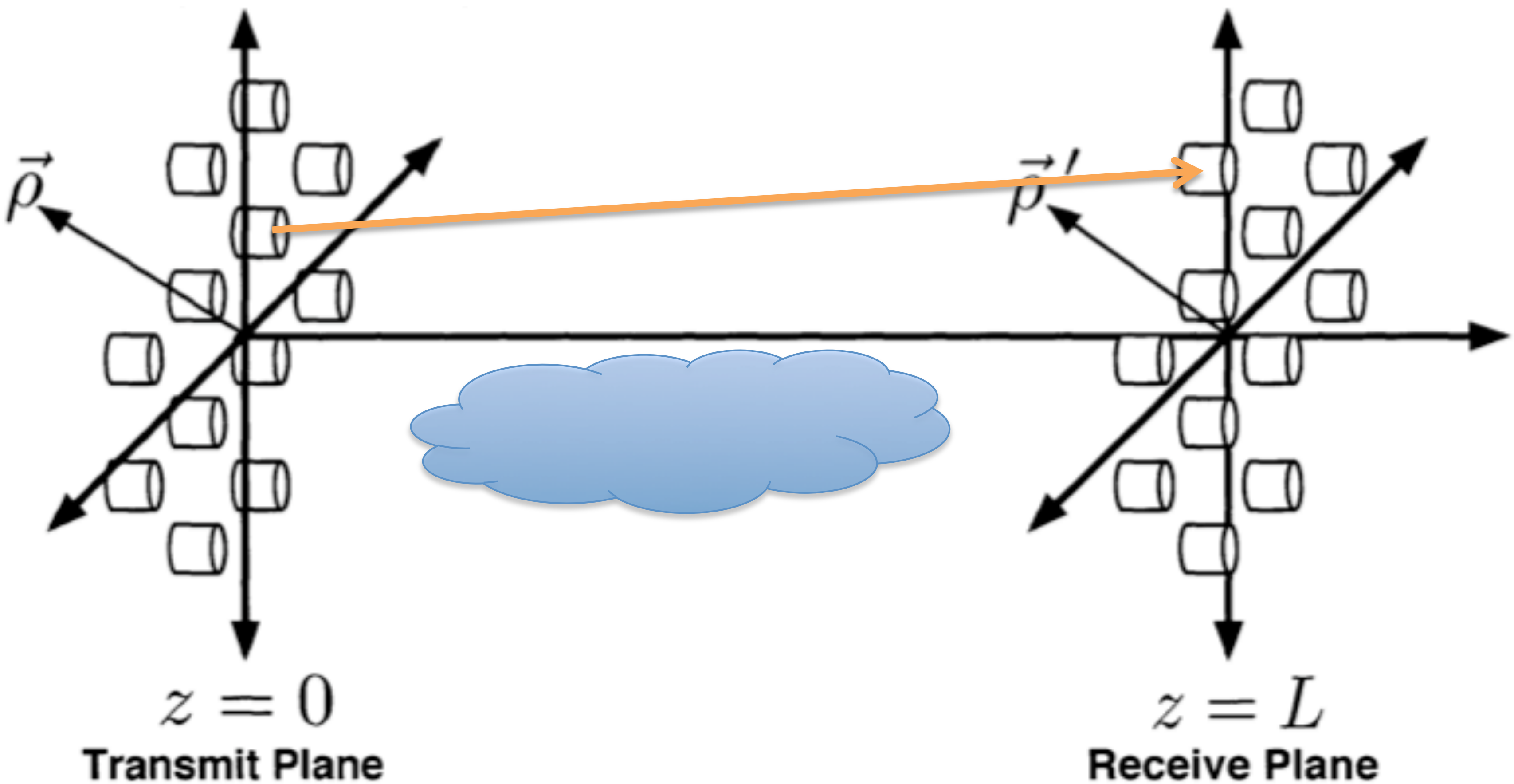
Spatial Diversity via Sparse Apertures



Spatial Diversity via Sparse Apertures



We can improve performance with **wavefront predistortion**.

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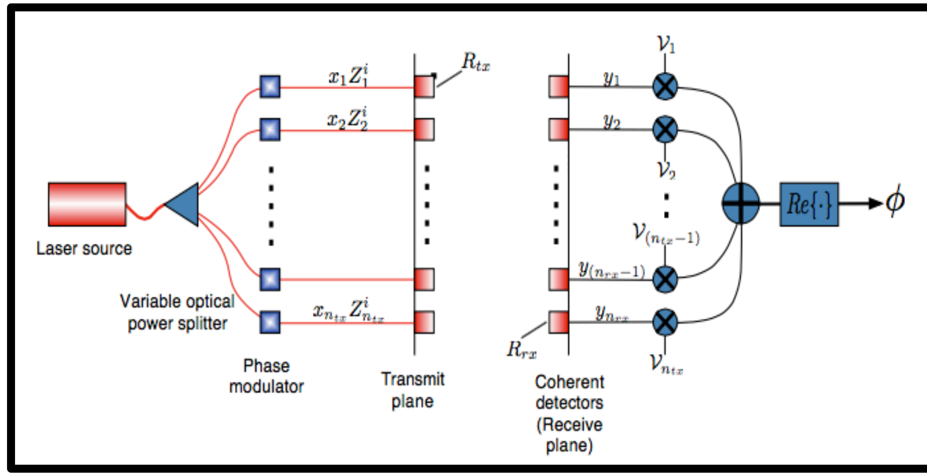
Channel Formulation

$$\vec{y} = \sqrt{\frac{\text{SNR}}{n_{rx}}}\mathbf{H}\vec{x} + \vec{w}$$



n_{tx} : number of transmit apertures

n_{rx} : number of receive apertures

$\vec{x}$: amplitude and phase of the output field at the transmit aperture

$\vec{y}$: amplitude and phase of the received field at each receive aperture

$\mathbf{H}$: channel transfer matrix, with element h_{xy} representing the diffraction gain of the field from transmit aperture x to receive aperture y

SNR : signal-to-noise ratio for a single aperture transmitter to a single aperture receiver with no turbulence

$\vec{w}$: circularly symmetric complex AGWN, unit variance

Image source: Puryear, 2011

Channel Formulation

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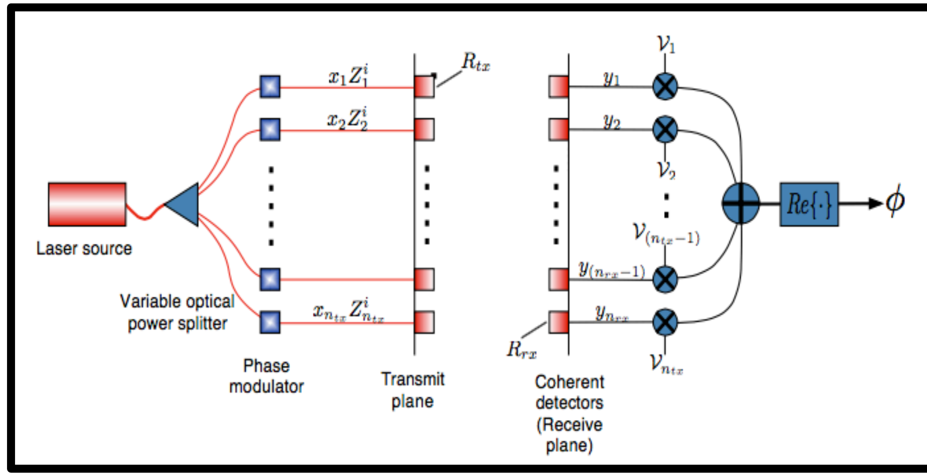

$$h_{mn} = \underbrace{e^{\chi(\rho'_m,\rho_n)+j\phi(\rho'_m,\rho_n)}}_{\text{Turbulence Fading Factor}} \underbrace{e^{jk\frac{|\rho'_m-\rho_n|^2}{2L}} e^{jkL+j\frac{\pi}{2}}}_{\text{Direct Path Gain}}$$

Turbulence Fading Factor

Direct Path Gain

Channel Formulation

$$\chi(\rho'_m, \rho_n) \sim \mathcal{N}(m_\chi, \sigma^2_\chi)$$

$$\mathbb{E}\left[e^{\chi 2} = 1\right] \Rightarrow m_\chi = -\sigma^2_\chi$$

$$\phi(\rho'_m, \rho_n) \sim \mathcal{N}(m_\phi, \sigma^2_\phi)$$

$$\sigma^2_\phi \gg 2\pi$$

$$\phi(\rho'_m, \rho_n) \sim \text{Unif}\left[0, 2\pi\right]$$

$$h_{mn} = e^{\chi(\rho'_m, \rho_n) + j\phi(\rho'_m, \rho_n)} e^{jk\frac{|\rho'_m - \rho_n|^2}{2L}} e^{jkL + j\frac{\pi}{2}}$$

Turbulence Fading Factor

Direct Path Gain

Channel Formulation

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$$h_{mn} = e^{\chi(\rho'_m, \rho_n) + j\phi(\rho'_m, \rho_n)}$$

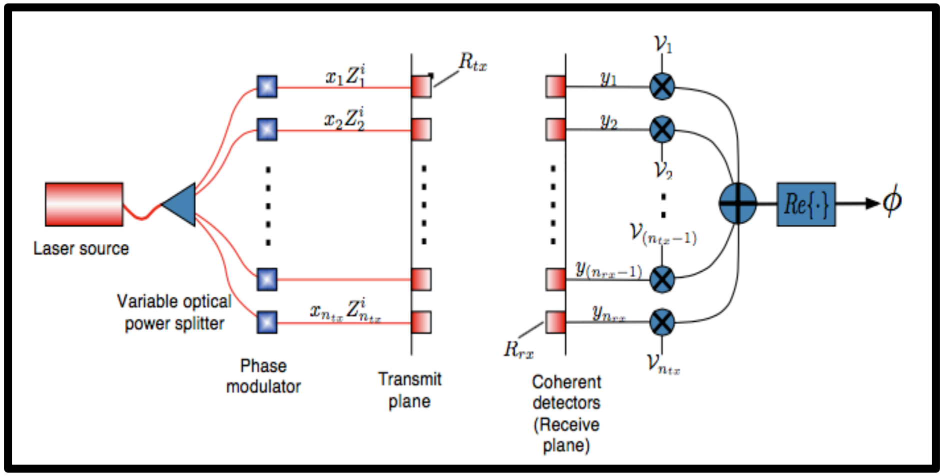
Turbulence
Fading Factor

Channel Formulation

- N_{tx} x N_{rx} i.i.d. entries of H
  - *Independent*: Transmit and receive apertures are separated by atmospheric correlation length.
  - *Identical*: Path distance is much greater than distance between apertures.
  - *Time-invariant*: bit period is much less than atmospheric correlation time.

Channel Formulation

$$\frac{1}{\sqrt{n_{rx}}}\mathbf{H} = \mathbf{U}\mathbf{\Gamma}\mathbf{V}^{\dagger}$$



$\mathbf{U}$: The i^{th} column $\vec{u_i}$ is the i^{th} output spatial eigenmode.

$\mathbf{V}$: The i^{th} column $\vec{v_i}$ is the i^{th} input spatial eigenmode.

$\mathbf{\Gamma}$: Diagonal matrix of singular values γ of $\mathbf{H}$.

γ_i^2 is the diffraction gain of the i^{th} spatial eigenmode.

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Randomly Generate Channel Transfer Matrix

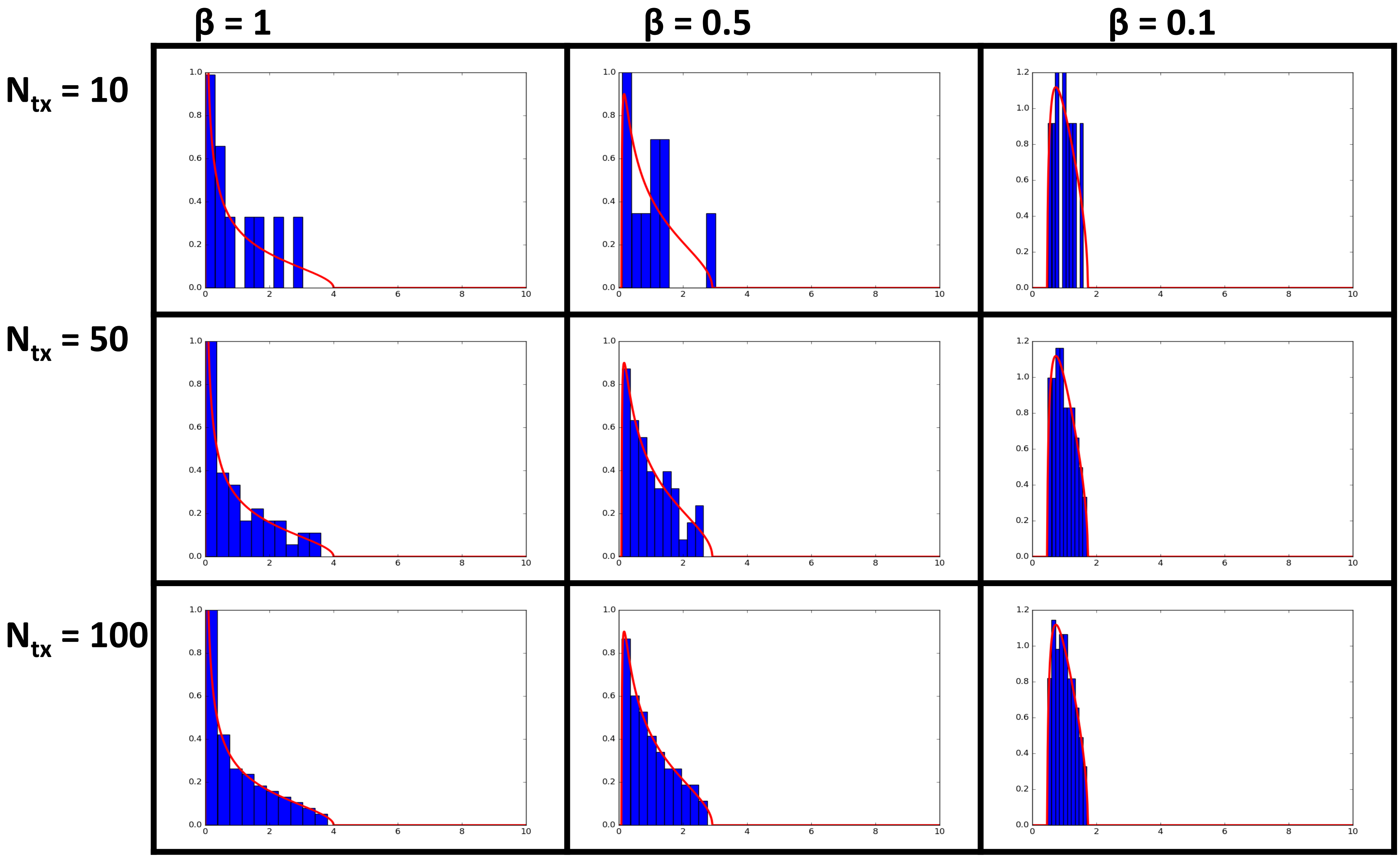
```
function generateH(N_rx,N_tx,L,C_n2)
    ##Log-amplitude fluctuations
    varX = minimum([0.124*k^(7/6)*C_n2*L^(11/6), 0.5]);
    mX = -varX;
    Z = randn(N_rx,N_tx);
    X = e.^(Z*sqrt(varX) + mX);

    ##Log-phase fluctuations
    phi = rand(N_rx,N_tx) * 2 * pi;

    ##Channel Transfer Matrix
    H = X.*e.^(im*phi);
end
```

Compare Squared Singular Values to Marcenko-Pastur Distribution

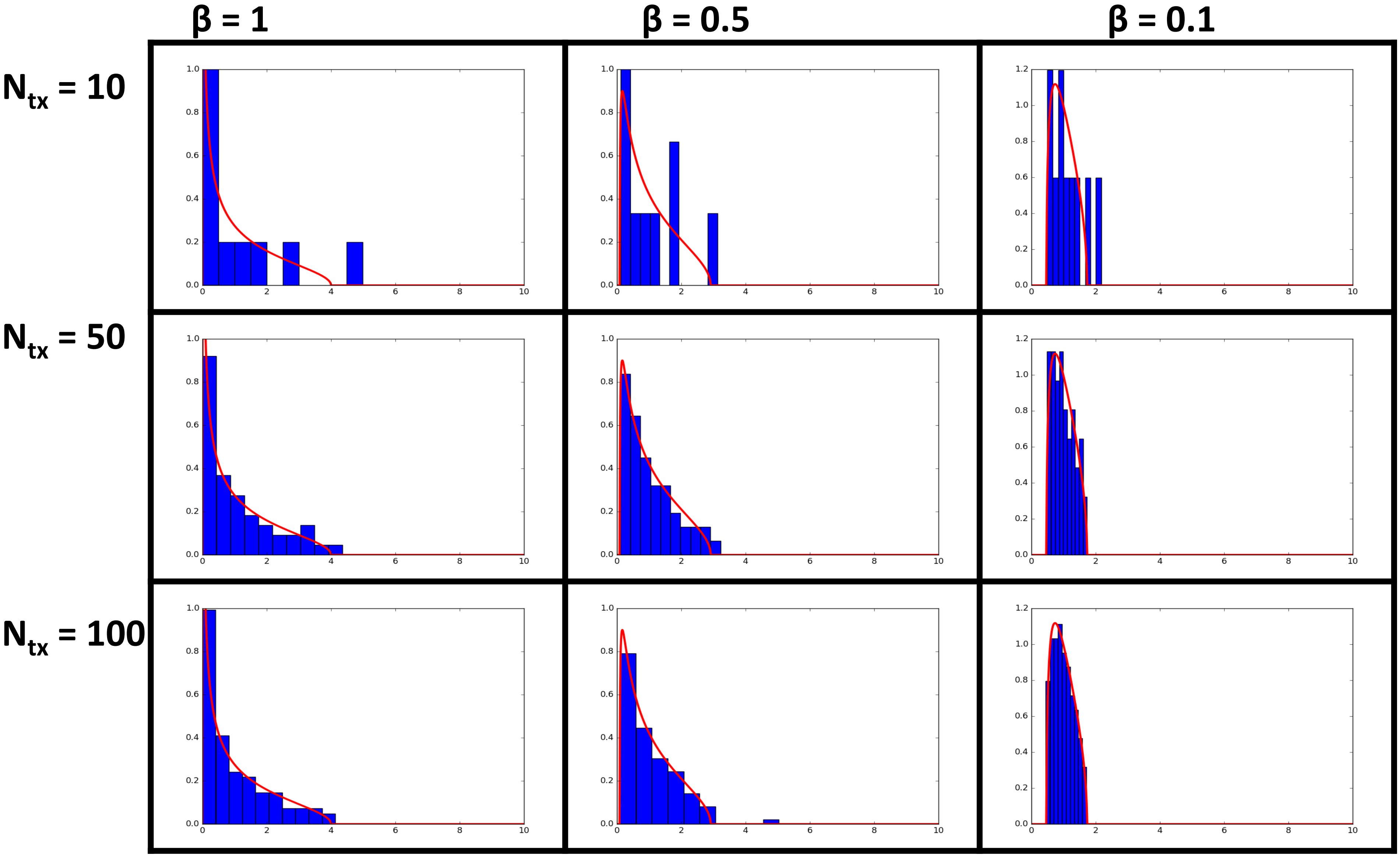
```
function diffGain(N_rx, N_tx, L, C_n2)

    H = generateH(N_rx,N_tx,L,C_n2);
    A = H*H'/N_rx;
    gamma = eigvals(A);

end

Beta = N_tx/N_rx;
```

```
function M_P(x, Beta)
    if x == 0
        ##f = maximum([0; 1 - Beta]);
        f=0;
    else
        num1 = maximum([0;(x-(1-
                sqrt(Beta))^2)]);
        num2 = maximum([0;
                ((1+sqrt(Beta))^2-x)]);
        f = sqrt(num1*num2)./
            (2*pi*x*Beta);
    end
end
```

Weak Turbulence



Strong Turbulence



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Minimum Average Bit Error Rate

Assume we know the instantaneous atmospheric state.

Then we minimize the BPSK bit error rate by choosing:

$$\vec{x} = a\vec{v}_{\max}$$
$$a = \{-1, 1\}$$

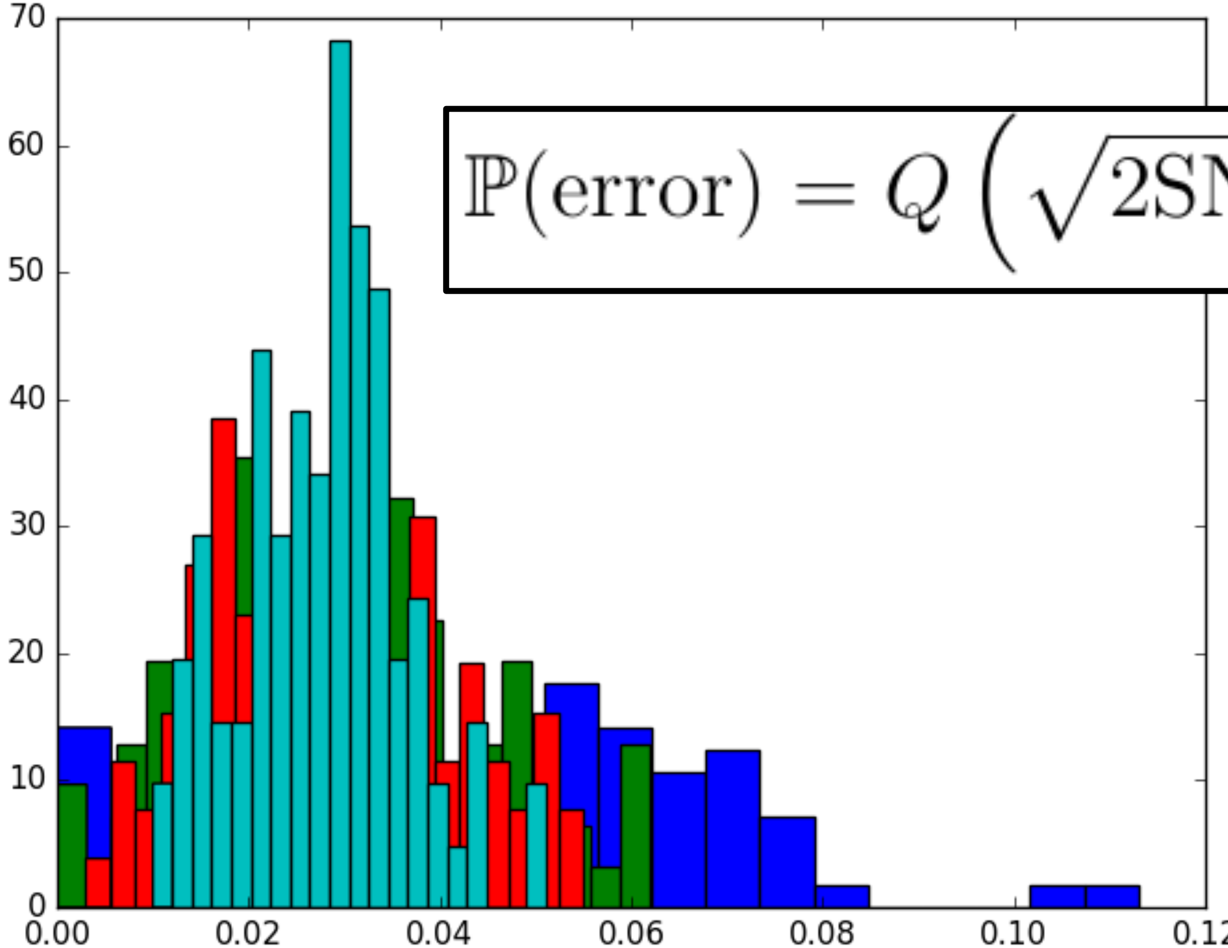
A sufficient detection statistic is:

$$\phi = Re\{\vec{u}^{\dagger}_{\max}\vec{y}\}$$

The largest square singular value converges to:

$$\gamma^2_{\max} = (1 + \sqrt{\beta})^2$$

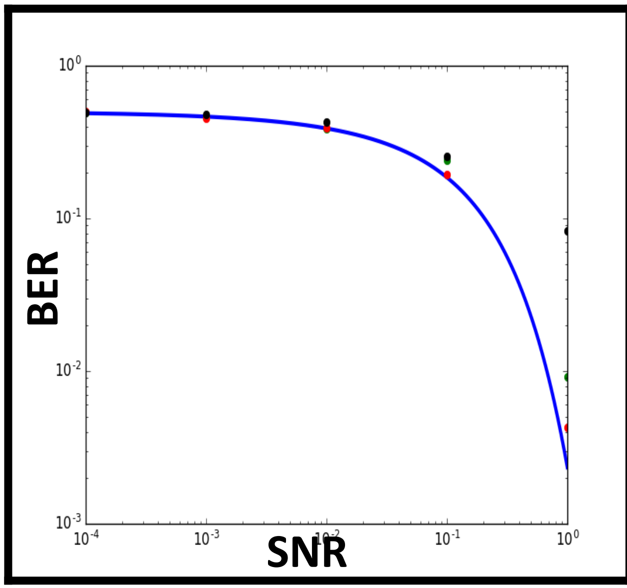
PDF of Bit Error Rate ←→Tracy-Widom



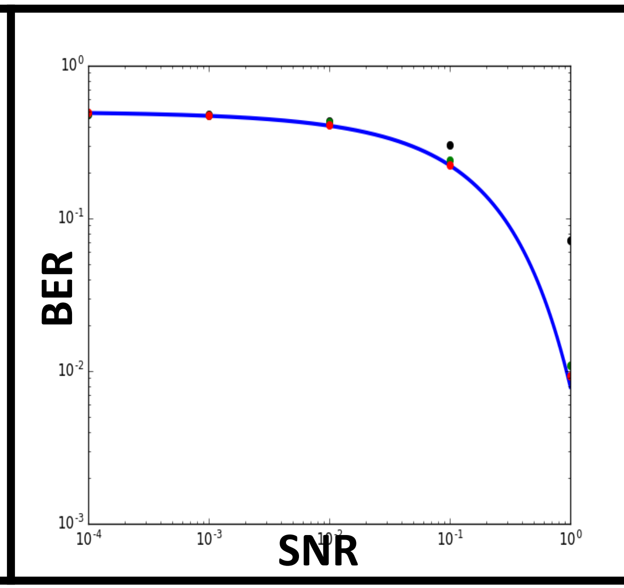
$$\mathbb{P}(\text{error}) = Q\left(\sqrt{2\text{SNR}\gamma_{\max}^2}\right)$$

Convergence of Bit Error Rate

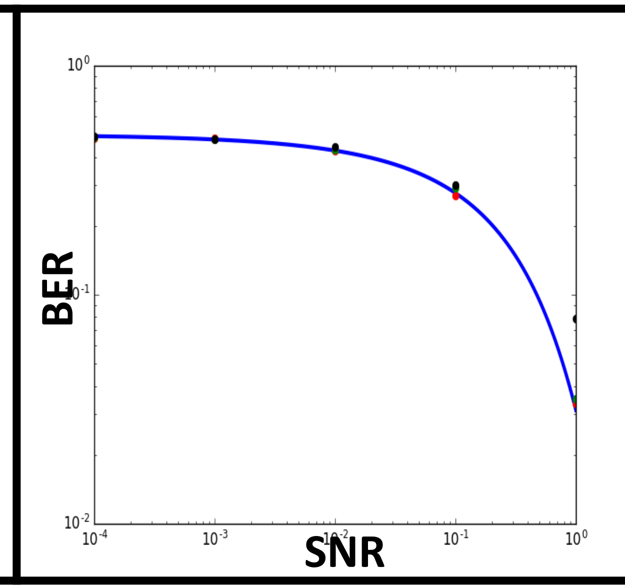
β = 1



β = 0 .5



β = 0.1



$$\lim_{n_{rx}\to\infty} \mathbb{E}\left[\mathbb{P}(\text{error})\right] = Q\left(\sqrt{2\text{SNR}\left(1+\sqrt{\beta}\right)^2}\right)$$

Convergence of Bit Error Rate

β = 1