Section 5-4

The Coordinate Ring of an Affine Variety

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In this section, we will use the results from Sections 5-2 and 5-3 to study the ring $k[V]$ of polynomial functions on an affine variety $V \subset k^n$.

Recall that $k[V]$ is defined as the collection of polynomial functions $\phi : V \to k$.

We will identify $k[V]$ with the quotient ring $k[x_1, ..., x_n]/\mathbf{I}(V)$, using the isomorphism

$$k[V] \cong k[x_1, ..., x_n]/\mathbf{I}(V)$$

Given a polynomial $f \in k[x_1, ..., x_n]$, we let $[f]$ denote the polynomial function in $k[V]$ represented by f.

Definition 1:

The *coordinate ring* of an affine variety $V \subset k^n$ is the ring $k[V]$.

Definition 2:

Let $V \subset k^n$ be an affine variety.

1. For any ideal $J = \langle \phi_1, ..., \phi_s \rangle \subset k[V]$, we define

$$\mathbf{V}_V(J) = \{(a_1, ..., a_n) \in V : \ \phi(a_1, ..., a_n) = 0$$
$$\textit{for all } \phi \in J\}.$$

We call $\mathbf{V}_V(J)$ a *subvariety* of V.

2. For each subset $W \subset V$, we define

$$\mathbf{I}_V(W) = \{\phi \in k[V] : \ \phi(a_1, ..., a_n) = 0$$
$$\textit{for all } (a_1, ..., a_n) \in W\}.$$

Example:

Let $V = \mathbf{V}(z - x^2 - y^2) \subset \mathbf{R}^3$.

Furthermore, let $J = \langle [x] \rangle \in \mathbf{R}[V]$. Then

$$W = \mathbf{V}_V(J) = \{(0, y, y^2) : \ y \in \mathbf{R}\} \subset V$$

is a subvariety of V.

Proposition 3:

Let $V \subset k^n$ be an affine variety.

1. For each ideal $J \subset k[V]$, $W = \mathbf{V}_V(J)$ is an affine variety in k^n contained in V.

2. For each subset $W \subset V$, $\mathbf{I}_V(W)$ is an ideal of $k[V]$.

3. If $J \subset k[V]$ is an ideal, then
$$J \subset \sqrt{J} \subset \mathbf{I}_V(\mathbf{V}_V(J)).$$

4. If $W \subset V$ is a subvariety, then
$$W = \mathbf{V}_V(\mathbf{I}_V(W)).$$

Proof:

1. By Proposition 10 of 5-2, there is a one-to-one correspondence between ideals of $k[V]$ and the ideals in $k[x_1, ..., x_n]$ containing $\mathbf{I}(V)$. Let

$$\widetilde{J} = \{f \in k[x_1, ..., x_n] : \ [f] \in J\} \subset k[x_1, ..., x_n]$$

be the ideal corresponding to $J \subset k[V]$.

Now, because $\mathbf{I}(V) \subset \widetilde{J}$, we have that $\mathbf{V}(\widetilde{J}) \subset V$.

The elements of $\widetilde{J}$ represent functions in J on V, so $\mathbf{V}(\widetilde{J}) = \mathbf{V}_V(J)$ by definition.

Therefore, $W = \mathbf{V}_V(J)$ is an affine variety contained in k^n.

This proves Part 1. The proofs of remaining parts are similar to other arguments in previous chapters; the book leaves them as an exercise.

Proposition 4:

An ideal $J \subset k[V]$ is radical if and only if the corresponding ideal

$$\widetilde{J} = \{f \in k[x_1, ..., x_n] : \; [f] \in J\} \subset k[x_1, ..., x_n]$$

is radical.

Proof:

Suppose J is radical, and let $f \in k[x_1, ..., x_n]$ satisfy $f^m \in \widetilde{J}$ for some $m \geq 1$. Then

$$[f^m] = [f]^m \in J$$

It follows that $[f] \in J$, because J is radical. Therefore, $f \in \widetilde{J}$, which means $\widetilde{J}$ is also radical.

On the other hand, if $\widetilde{J}$ is radical and $[f]^m \in J$, it follows that $[f^m] \in J$, and hence $f^m \in \widetilde{J}$.

Therefore, $f \in \widetilde{J}$, because $\widetilde{J}$ is radical. Then $[f] \in J$, which proves J is radical.

Theorem 5:

Let k be an algebraically closed field and let $V \subset k^n$ be an affine variety.

1. The **Nullstellensatz** in $k[V]$: If J is any ideal in $k[V]$, then

$$\mathbf{I}_V(\mathbf{V}_V(J)) = \sqrt{J} = \{[f] \in k[V] : [f]^m \in J\}.$$

2. The correspondences

$$\begin{array}{ccc} W \subset V & \mathbf{I}_V & \\ \{\ \textit{affine subvarieties}\ \} & \rightleftarrows & \{\ \textit{radical ideals}\ \} \\ & \mathbf{V}_V & J \subset k[V] \end{array}$$

are inclusion-reversing bijections and are inverses of each other.

3. Under the correspondence given above, points of V correspond to maximal ideals of $k[V]$.

Proof:

1. Let J be an ideal of $k[V]$. By Prop. 10 of 5-2, J corresponds to $\widetilde{J} \subset k[x_1, ..., x_n]$ as in Prop. 3 of this section.

Then $\mathbf{V}(\widetilde{J}) = \mathbf{V}_V(J)$. Therefore,

$$\textit{If } [f] \in \mathbf{I}_V(\mathbf{V}_V(J)), \textit{ then } f \in \mathbf{I}(\mathbf{V}(\widetilde{J})).$$

We apply the Nullstellensatz in k^n, and we have that $\mathbf{I}(\mathbf{V}(\widetilde{J})) = \sqrt{\widetilde{J}}$.

This means that $f \in \sqrt{\widetilde{J}}$, and by definition $f^m \in \widetilde{J}$ for some $m \geq 1$.

Therefore, $[f^m] = [f]^m \in J$, and as a result $[f] \in \sqrt{J}$ in $k[V]$.

This proves $\mathbf{I}_V(\mathbf{V}_V(J)) \subset \sqrt{J}$.

Now, by Part 3 of Prop. 3, $\sqrt{J} \subset \mathbf{I}_V(\mathbf{V}_V(J))$ holds for any ideal $J \subset k[V]$.

We conclude that $\mathbf{I}_V(\mathbf{V}_V(J)) = \sqrt{J}$, and the proof is complete.

2. If J is a radical ideal, then $J = \sqrt{J}$. By Part 1, $\mathbf{I}_V(\mathbf{V}_V(J)) = \sqrt{J}$, which implies $\mathbf{I}_V(\mathbf{V}_V(J)) = J$.

At the same time, by Prop. 3, Part 4, we know that for any subvariety $W \subset V$ it is true that $W = \mathbf{V}_V(\mathbf{I}_V(W))$.

Therefore $\mathbf{V}_V$ and $\mathbf{I}_V$ are inverses of each other. The fact that they are inclusion-reversing can be shown as in Chap. 4.

3. Proof uses same method as for Thm. 11 of Section 4-5.

Definition 6:

Let $V \subset k^m$ and $W \subset k^n$ be affine varieties.

V and W are said to be *isomorphic* if there exist polynomial mappings

$$\alpha: \; V \rightarrow W \; and \; \beta: \; W \rightarrow V$$

such that

$$\alpha \circ \beta = id_W \; and \; \beta \circ \alpha = id_V.$$

Here, id_V is the identity mapping from V to itself.

Example 7:

Consider the following surfaces in $\mathbf{R}^3$:

$$Q_1 = \mathbf{V}(x^2 - xy - y^2 + z^2) = \mathbf{V}(f_1)$$
$$Q_2 = \mathbf{V}(x^2 - y^2 + z^2 - z) = \mathbf{V}(f_2)$$

We wish to examine the intersection curve $C = \mathbf{V}(f_1, f_2)$ of the two surfaces.

At first glance, C is not easy to visualize. To make our task easier, we observe that

$C = \mathbf{V}(f_1, f_2) = \mathbf{V}(f_1, f_1 + cf_2)$, where $c \in \mathbf{R}, c \neq 0$.

Since $\mathbf{V}(f_1, f_1 + cf_2) \subset \mathbf{V}(f_1 + cf_2)$,

$$C \subset \mathbf{V}(f_1 + cf_2).$$

Let $F = \mathbf{V}(f_1 + cf_2)$ and set $c = -1$. Then

$$F = \mathbf{V}(f_1 + cf_2) = \mathbf{V}(f_1 - f_2) = \mathbf{V}(z - xy).$$

The surface F is isomorphic as a variety to $\mathbf{R}^2$. The polynomial mappings are the following:

$$\alpha : \; \mathbf{R}^2 \longrightarrow F,$$
$$(x, y) \mapsto (x, y, xy),$$
$$\pi : F \longrightarrow \mathbf{R}^2,$$
$$(x, y, z) \mapsto (x, y)$$

These mappings satisfy $\alpha \circ \pi = id_F$ and $\pi \circ \alpha = id_{\mathbf{R}^2}$.

To visualize C, we can project to the curve $\pi(C) \subset \mathbf{R}^2$.

The equation for $\pi(C)$ is

$$x^2y^2 + x^2 - xy - y^2 = 0,$$

obtained by substituting $z = xy$ in either f_1 or f_2.

Each point (a, b) on $\pi(C)$ corresponds to point (a, b, ab) on C.

Proposition 8:

Let V and W be varieties.

1.

Let $\alpha : V \to W$ be a polynomial mapping. Then for every polynomial function $\phi : W \to k$, the composition $\phi \circ \alpha : V \to k$ is also a polynomial function.

In addition, the map $\alpha^* : k[W] \to k[V]$ defined by $\alpha^*(\phi) = \phi \circ \alpha$ is a ring homomorphism which is the identity on the constant functions $k \subset k[W]$.

Note: α^* is often called the *pullback mapping* on functions because it goes in the opposite direction from α.

2.
Let $f : k[W] \to k[V]$ be a ring homomorphism which is the identity on constants.

Then there is a unique polynomial mapping $\alpha : \ V \to W$ such that $f = \alpha^*$.

Proof:

Let $x_1, ..., x_m$ be the coordinates of $V \subset k^m$, and $y_1, ..., y_n$ be the coordinates of $W \subset k^n$.

Then $\phi : W \to k$ can be written as a polynomial $f(y_1, ..., y_n)$.

Also, $\alpha : V \rightarrow W$ can be written as an n-tuple of polynomials:

$$\alpha(x_1, ..., x_m) = (a_1(x_1, ..., x_m), ..., a_n(x_1, ..., x_m)).$$

Substituting $\alpha(x_1, ..., x_m)$ into ϕ, we obtain

$$(\phi \circ \alpha)(x_1, ..., x_m) = f(a_1(x_1, ..., x_m), ..., a_n(x_1, ..., x_m)).$$

This is a polynomial in $x_1, ..., x_m$, so $\phi \circ \alpha$ is a polynomial function on V.

We can define $\alpha^* : k[W] \to k[V]$ using the expression $\alpha^*(\phi) = \phi \circ \alpha$.

Now we want to show that α^* is a ring homomorphism. Consider another element ψ of $k[W]$, represented by $g(y_1, ..., y_n)$. Then

$$\alpha^*(\phi+\psi) = f(a_1, ..., a_n)+g(a_1, ..., a_n) = \alpha^*(\phi)+\alpha^*(\psi)$$

The condition $\alpha^*(\phi \cdot \psi) = \alpha^*(\phi) \cdot \alpha^*(\psi)$ is proved in a similar manner.

Therefore, sums and products are preserved, and α^* is a ring homomorphism.

Now we want to show that α^* is the identity on constant functions. Consider $[a] \in k[W]$ for some $a \in k$.

$[a]$ is a constant function on W with value a, which implies $\alpha^*([a]) = [a] \circ \alpha$ is constant on V (value a as well).

Therefore, $\alpha^*([a]) = [a]$, and α^* is the identity on constant functions.

This completes the proof of Part 1 of Prop. 8.

The proof of the second part is quite long, we won't have time to go through it here. It is explained in detail on pgs. 243-244 of the textbook.

Theorem 9:

Two affine varieties $V \subset k^m$ and $W \subset k^n$ are isomorphic if and only if there is an isomorphism $k[V] \cong k[W]$ of coordinate rings which is the identity on constant functions.

Sources Used:

Ideals, Varieties, and Algorithms, by Cox, Little, O'Shea; UTM Springer, 3rd Ed., 2007.

Thank You!