THE SHANNON TRANSFORM IN RANDOM MATRIX THEORY

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The landmark contributions to the theory of random matrices of Wishart (1928), Wigner (1955), and Marčenko and Pastur (1967), were motivated to a large extent by their applications. In this paper we report on a new transform motivated by the application of random matrices to various problems in the information theory of noisy communication channels.

The Shannon transform of a nonnegative random variable X is defined as

$$\mathcal{V}_X(\gamma) = E[\log(1 + \gamma X)]. \tag{1}$$

where γ is a nonnegative real number. Originally introduced in [1-2], its applications to random matrix theory and engineering applications have been developed in [3]. In this paper we give a summary of its main properties and applications in random matrix theory. As is well known since the work of Marčenko and Pastur [4], it is rare the case that the limiting empirical distribution of the squared singular values of random matrices (whose aspect ratio converges to a constant) admit closed-form expressions. However, [4] showed a very general result where the characterization of the solution is accomplished through a fixed-point equation involving the Stieltjes transform. Also motivated by applications, [2] introduced the η-transform which is very related to both the Stieltjes and Shannon transforms and leads to compact definitions of other transforms used in random matrix theory such as the S-transform [5].

In applications in information theory, the Shannon transform is directly of interest as it gives the capacity of various noisy coherent communication channels. In the paper we give several examples of closed-form expressions for the Shannon transform, including the Marčenko-Pastur law, and instances where the Shannon and η transforms lead to particularly simple solutions for the limiting empirical distribution of the squared singular values of random matrices with dependent entries.

References

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