

Department of Mechanical Engineering  
Massachusetts Institute of Technology  
2.010 Modeling, Dynamics and Control III  
Spring 2002

**SOLUTIONS:** Problem Set # 10

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**Problem 1**

Estimating transfer function from Bode Plot.

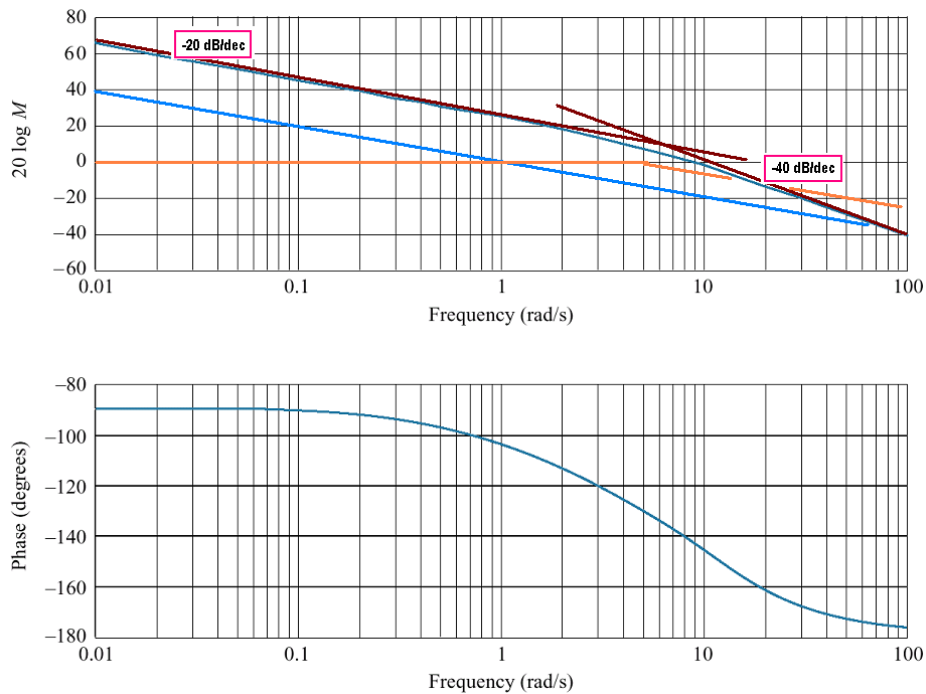
This is something that is very useful when working with experimental data. It comes with practice. If you see how to do it immediately, great, if not here are some tips on how to do it a little bit more methodological. It's long but hopefully worthwhile.

**How to obtain a transfer function from the graph**  
(Melissa's Way, not necessarily the best way)

**Step 1: Relative order of system**

Look at Phase diagram. The final value will tell you the relative order of the system. Think about it. Each pole at high frequencies will contribute  $-90^\circ$  to the phase, while each zero will contribute  $+90^\circ$ . (Complex poles/zeros are made of 2 poles/zeros so they contribute  $-/+180^\circ$ ) You can also do this by looking at the final slope of the magnitude response, but I think it is clearer to look at the phase response.

Now for this problem we see that the phase diagram ends at  $-180^\circ$ . This tells you that there are 2 more poles than zeros in the transfer function.



## Step 2: Number of Poles and zeros

Now we need to decide how many poles and zeros there are in the system. Looking at the magnitude diagram we can see that the initial slope is  $-20$  dB/dec, and the final slope is  $-40$  dB/dec. Since we never see a decrease in slope i.e. from  $-40$  dB/dec to  $-20$  dB/dec, we can assume that there are no zeros.

So we know we have two poles and no zeros.

## Step 3: Location of Poles and zeros

Since the magnitude plot starts at a slope of  $-20$  dB/dec we know that this is a type 1 system, so we have a pole at 0. We can also see this from the phase diagram since it starts at  $-90^\circ$ .

Now we can figure out where the second pole is. By extending the asymptotic lines at the beginning and end of the diagram we can see that they meet at about 5 rad/s. Therefore this is the break frequency of second pole.

Now we know our transfer function looks like:

$$G(s) = \frac{K}{s\left(\frac{s}{5} + 1\right)}$$

The last thing we need to do is find the gain that gives us this response

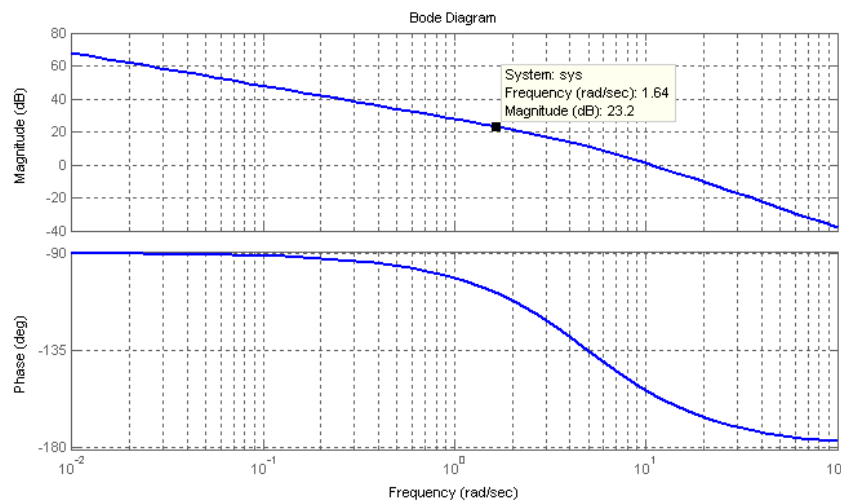
## Step 4: Finding the Gain

Since this is a Type 1 system, the gain will not simply be the point where it crosses as it would for a Type 0 system. Instead we can figure out the gain by knowing that the pole at 0 would be at  $+40$  dB if there were no gain. Instead we can see that it is more at about 68 dB. Therefore the DC gain must have contributed this additional 28 dB. The gain is then :  
 $20\log K = 28\text{dB} \Rightarrow K = 25$

$$G(s) = \frac{25}{s\left(\frac{s}{5} + 1\right)} = \frac{125}{s(s+5)}$$

## Step 5: MATLAB Check

To verify that this is the correct we can plot it into MATLAB.

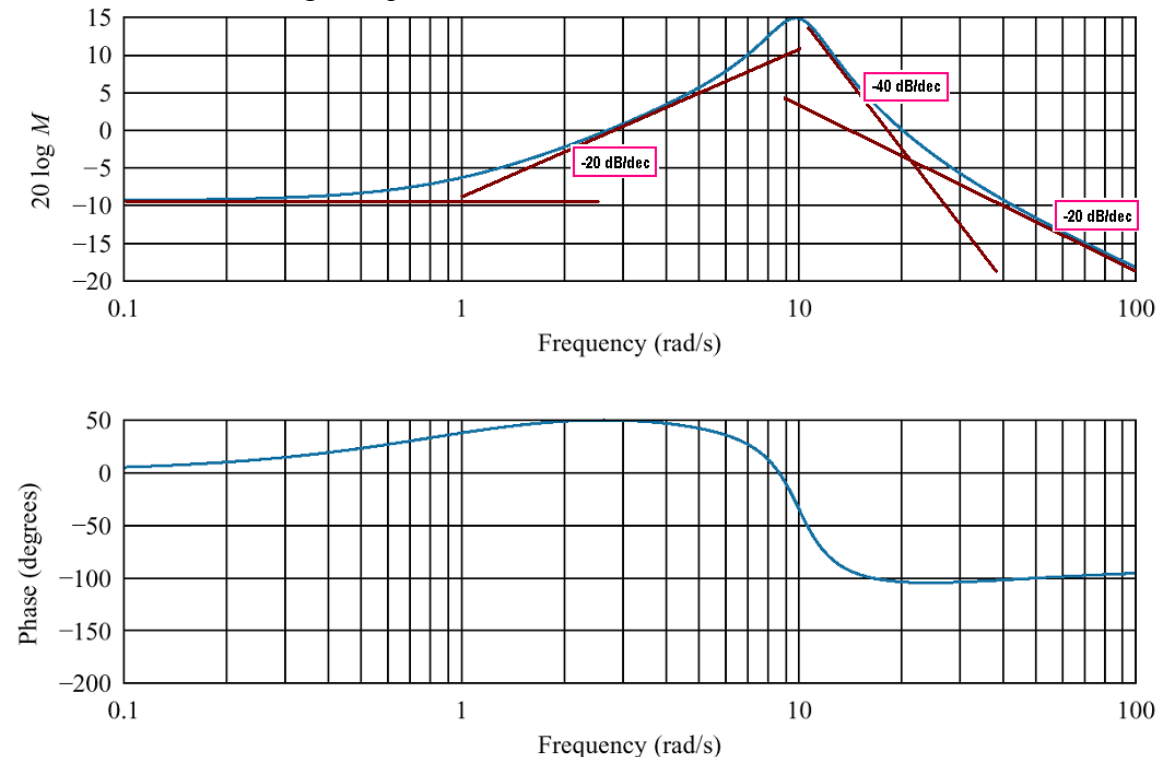


Indeed the response is very close to our original function.

## Part b) – same method

### Step 1: Relative order of system

Here we see that the phase plot starts at  $0^\circ$  and drops down to about  $-90^\circ$ , which implies a relative order of 1. Notice that even though the phase goes up to about  $50^\circ$ , what we care about is the final value for high frequencies.



### Step 2: Number of Poles and zeros

Now we know that we have one more pole than zero, but we don't know how many or where they are. We need to carefully analyze this system. Again we look at the slopes of the response. Initially the slope is 0, so we know we have a Type 0 system. Then the response goes up, at a slope of +20 dB/dec; so there must be a zero around 1 rad/s. Around 10 rad/s the response drops and goes down at a slope of about -40 dB/dec. Which means that there must have been a contribution of 3 poles. One to offset the zero, and 2 more bring the slope down to -40. Then around 20 rad/s we see another change in slope from -40 dB/dec to -20 dB/dec, which must be due to another zero.

Therefore we can safely say that we have 3 poles and 2 zeros, with a relative order of 1 as expected.

### Step 3: Location of Poles and zeros

By drawing the asymptotes as shown in the figure we can estimate the break frequency for the first zero at 1 rad/s. We see though that the response increases beyond the expected slope of the asymptote. The only other thing that would cause the response to rise other than a zero is a second order pole with  $\zeta < 0.707$ . So then we know that around a frequency of 10 rad/s we have a complex double pole. In addition around this area we have an additional simple pole, since the slope between 10 – 20 rad/s is -40 dB/dec. This pole probably lies somewhere between 6 to 20 rad/s. Then the zero that cause the reduction in slope lies around 25 rad/s. So our transfer

function looks like:

$$G(s) = \frac{K \left( \frac{s}{1} + 1 \right) \left( \frac{s}{25} + 1 \right)}{\left( \frac{s}{10} + 1 \right) \left( \left( \frac{s}{10} \right)^2 + 2\zeta \left( \frac{s}{10} \right) + 1 \right)}$$

Where we still need to find K and  $\zeta$ .

#### Step 4: Finding the Gain

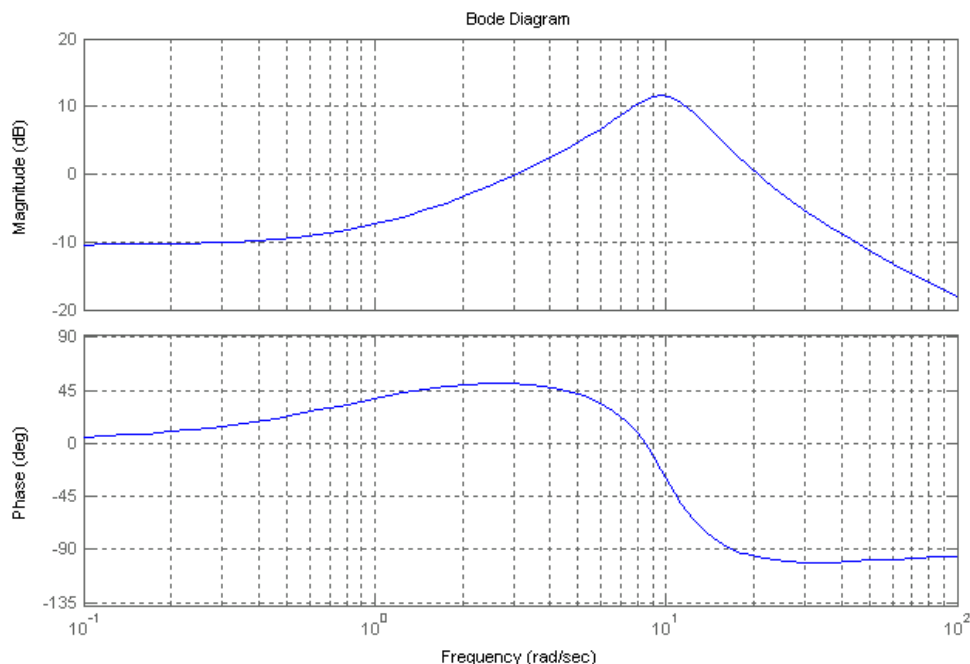
The gain for this problem will be the y-intercept at low frequencies so:  
 $20 \log K = -10 \text{ dB} \Rightarrow K = 0.32$ .

Since the second order pole causes a slight increase in the response but doesn't cause it to shoot up, we can estimate  $0.4 \leq \zeta < 0.707$ , we will choose  $\zeta = 0.6$  and check whether this is viable.

Finally we have

$$G(s) = \frac{0.32 \left( \frac{s}{1} + 1 \right) \left( \frac{s}{25} + 1 \right)}{\left( \frac{s}{10} + 1 \right) \left( \left( \frac{s}{10} \right)^2 + 2 \cdot 0.6 \left( \frac{s}{10} \right) + 1 \right)} = \frac{12(s+1)(s+25)}{(s+10)(s^2+6s+100)}$$

#### Step 5: MATLAB Check

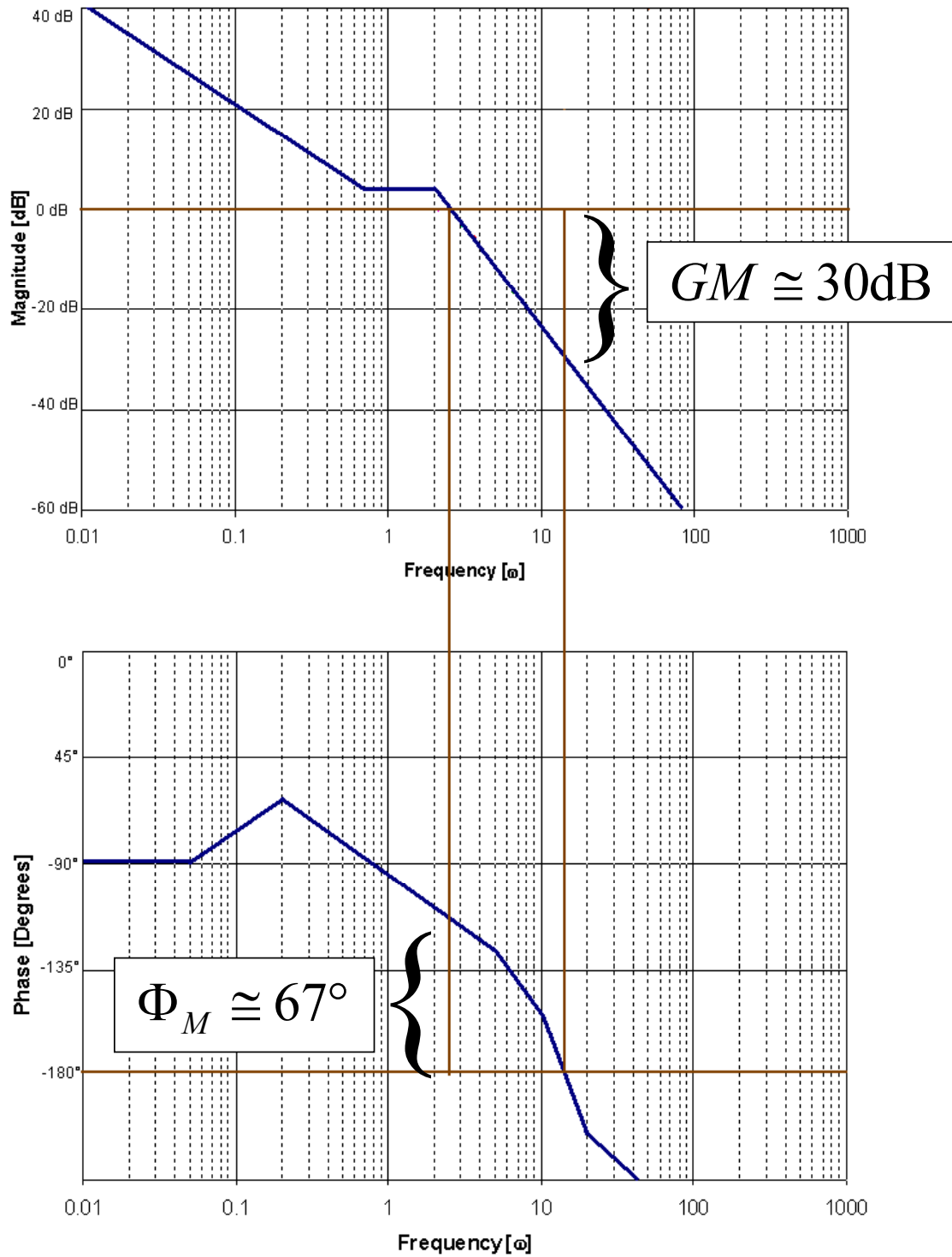


Again, good correlation with our matlab results

## Problem 2

You are asked to sketch the bode plot so it should be done by hand. When  $K = 800$ .

$$G(s) = \frac{K(s+0.5)}{s(s+100)(s+2)^2} = \frac{800\left(\frac{s}{0.5}+1\right)}{800s\left(\frac{s}{100}+1\right)\left(\frac{s}{2}+1\right)^2}$$



The gain margin of 30 dB implies that the entire magnitude curve can be shifted up by 30 dB, which is the equivalent of lowering the 0 dB mark by 30 dB. Since we can shift the curve by up to 30 dB that means that the gain can increase from what it is now by a factor of up to:  
 $20\log(\text{gain}) = 30 \text{ dB} \rightarrow \text{gain} = 31 \text{ times.}$

Therefore our value of K can be:

$$800 < K \leq 800 \cdot 31 = 24800$$

### Problem 3

This is a very simple problem as soon as you understand what you are looking at, and what you need to do.

a)

First let's remember how we obtain the gain margin, we find the point on the phase diagram where the phase is exactly  $180^\circ$  and then we find the magnitude at the same frequency and the distance from the 0dB is the gain margin.

In this problem we are not given a bode plot instead we are given the input/output curves –which is essentially how the bode plot is formed.

Since we are given input/output curves when the phase is  $180^\circ$ , all we need to do is find the gain at this frequency, which is:

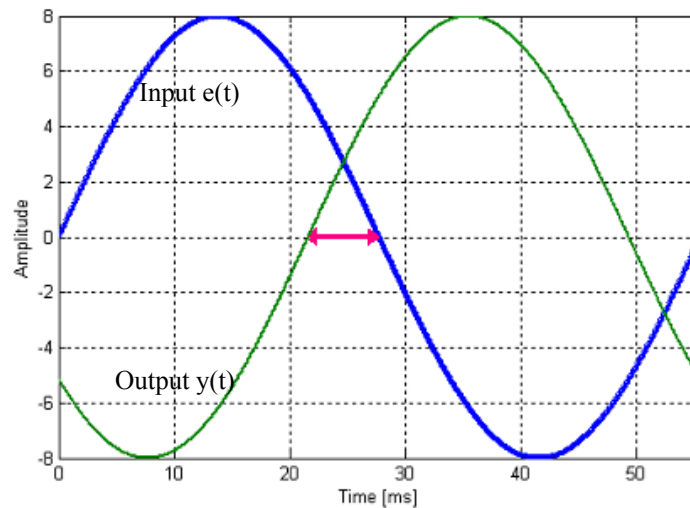
$$\frac{\text{output}}{\text{input}} = \frac{5}{8} \text{ or } 20\log\left(\frac{5}{8}\right) = -4.08 \text{ dB}$$

Therefore the gain margin is: 4.08 dB

b)

Now we are asked to find the phase margin.

With a bode plot we find the point where the response hits the 0 dB line and then find the phase at the same frequency. In figure (3d) we are given an input/output curve where the gain is 1 which is 0 dB. So, all we need to do is find out how much the sinusoid has been shifted.

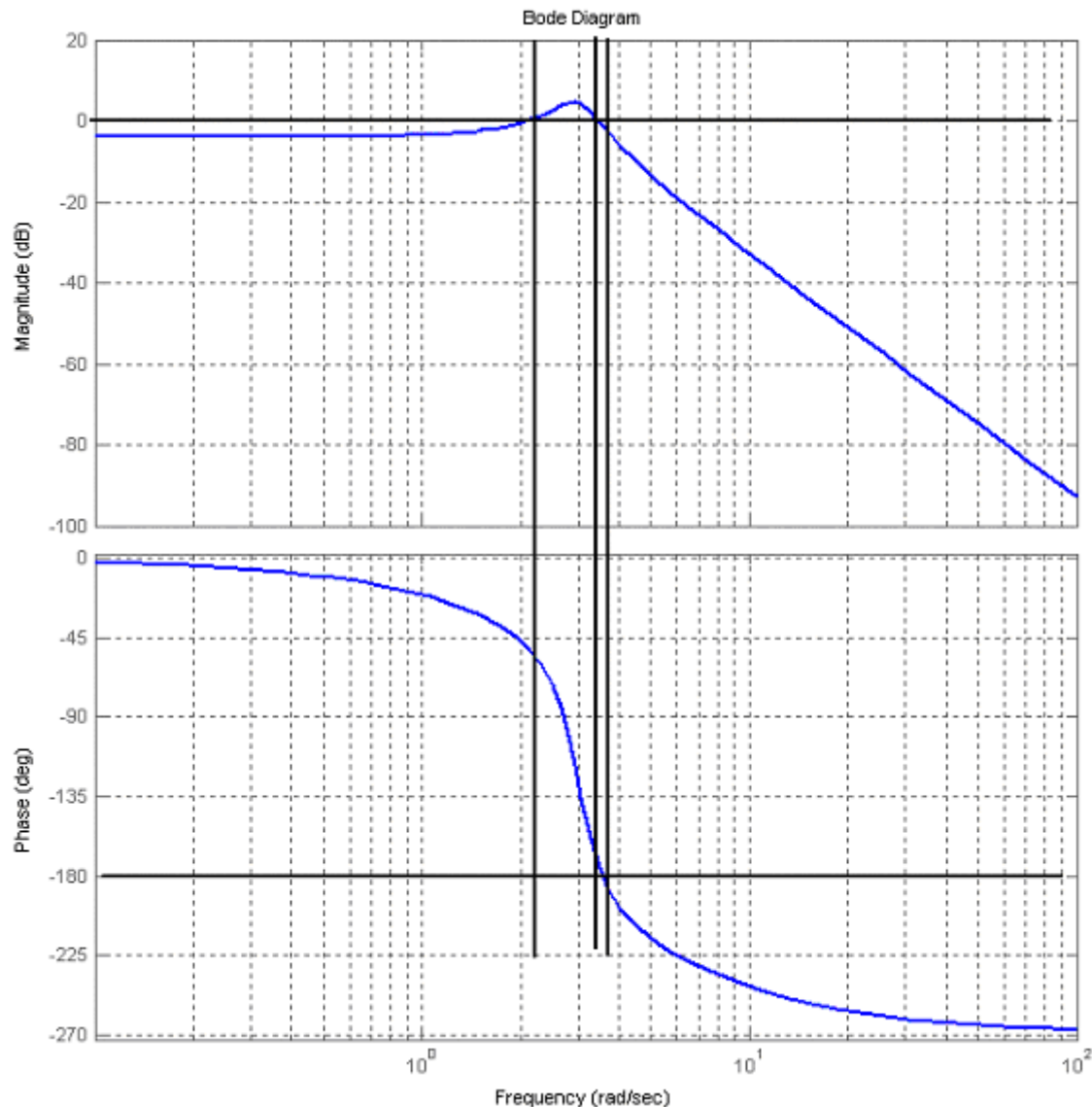


The distance shown is how much the output would need to shift to be at a phase of  $180^\circ$ , in other words this is our phase margin. By estimating this shift from the graph and converting it to degrees, we get that the shift margin is  $\Phi_M \cong 40^\circ$ .

#### Problem 4

You should not need to do this by hand, just use matlab. Here are the commands you need:

```
>> num = [22.5]
>> den = conv([1 4],[1 0.9 9])
>> bode(num,den)
```



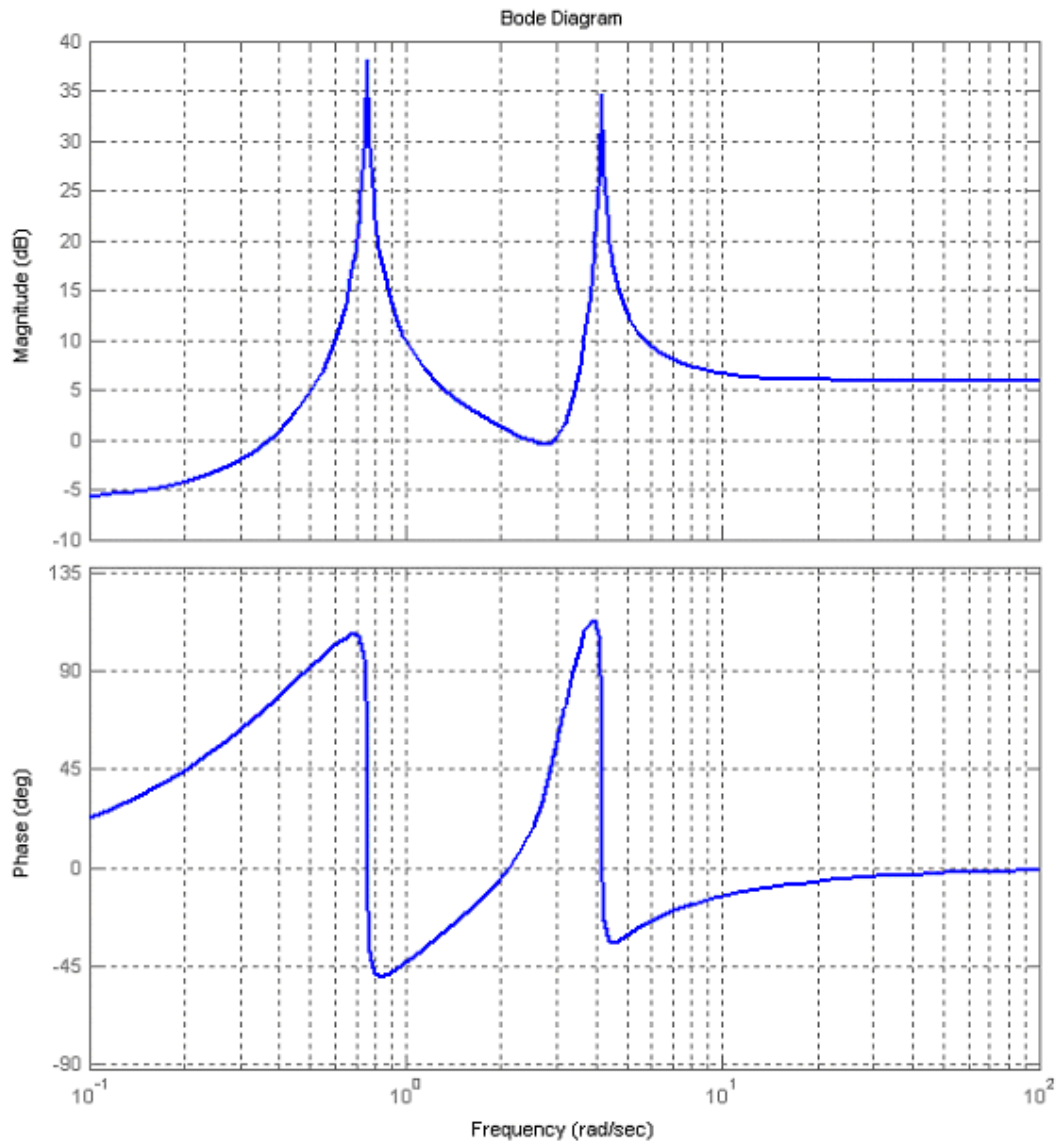
Though it is difficult to see using matlab we find that when the phase crosses  $180^\circ$  at  $\omega = 3.55$  rad/s the magnitude is  $-1.2$  dB, so the phase margin is  $1.2$  dB.

The magnitude will cross the 0 dB line twice the first time the phase about  $-50^\circ$ , which we dismiss. The second time the phase is about  $-178^\circ$ , so we say that the phase margin is  $2^\circ$ . This system is very close to instability.

### Problem 5

This one specially you should use matlab

```
>> num = 1.96*conv([1 1 .25],[1 1.26 9.87])  
>> den = conv ([1 .015 .57],[1 .083 17.2])  
>> bode (num,den)
```



If this is used as PID controller the frequencies reduced will be those of the peaks of the magnitude response. Namely  $\omega = 0.75$  rad/s and  $\omega = 4.1$  rad/s

If you plot this by hand, you MUST calculate the  $M_p$ , the peak of the magnitude, and  $\omega_r$  the frequency at which they occur.

b) These frequencies will be reduced because if the poles and zeros of the controller match the poles and zeros of the disturbance then there will be pole/zero cancellation and the dynamics of the disturbance will be reduced.