Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics and Control III
Spring 2002

SOLUTIONS: Problem Set #2

Problem 1

It seems a lot of you had problems with partial fractions. They are **FUNDAMENTAL** to this course, so you must understand them. I solved Problem 1a) and 1b) through different methods to illustrate the different ways of solving partial fractions. I hope this helps. Please look carefully at the solutions, or come to see me, if you don't understand them.

a)
Given:

$$\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+2\frac{dy}{dt}=2\frac{du}{dt}+3u$$

$$u = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

$$y(0)=1 \qquad \dot{y}(0)=-1 \qquad \ddot{y}(0)=2$$

Take the Laplace transform of the equation

$$s^3Y(s)-s^2y(0)-s\dot{y}(0)-\ddot{y}(0)$$
$$+3\left(s^2Y(s)-sy(0)-\dot{y}(0)\right)+2\left(sY(s)-y(0)\right)=2sU(s)+3U(s)$$

Substituting for the initial conditions and combining terms

$$\left[s^3+3s^2+2s\right]Y(s)-(s^2+2s+1)=(2s+3)U(s)$$

Obtain the time response y(t) by solving for the inverse Laplace

$$y(t)=L^{-1}\left\{G(s)U(s)\right\}+L^{-1}\left\{\text{initial conditions}\right\}$$

Now, since we are told that the input looks like a step function, we can substitute (1/s) for U(s).

$$y(t)=L^{-1}\left\{\frac{(2s+3)}{(s^3+3s^2+2s)}\frac{1}{s}\right\}+L^{-1}\left\{\frac{s^2+2s+1}{s^3+3s^2+2s}\right\}$$

BIG PICTURE: Why Partial Fractions?!
We need partial fractions because it is much easier to get the inverse Laplace of a first order (or second order as in 1b). Any fraction can be split up into a sum (or difference) of its factors.

The first term notice that we have a repeated root

$$\frac{(2s+3)}{s^2(s+1)(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} + \frac{D}{s+2} \qquad \text{(i)}$$

To systematically solve for the partial fraction multiply first by the highest term of the repeated root s^2 and evaluate at s = 0.

$$\frac{(2s+3)}{(s+1)(s+2)} = A + \frac{Bs^2}{s} + \frac{Cs^2}{s+1} + \frac{Ds^2}{s+2} \qquad \text{(ii)}$$

$$\frac{(3)}{(1)(2)} = A + \frac{B0^2}{s} + \frac{C0^2}{s+1} + \frac{D0^2}{s+2} \Rightarrow A = 3/2$$

To solve for B it is not possible to evaluate for s = 0 anymore, since that term disappears, instead we have to take the derivative of equation (ii). Use the quotient rule for the left hand side.

$$\frac{(2s+3)}{(s+1)(s+2)} = A + \frac{Bs^2}{s} + \frac{Cs^2}{s+1} + \frac{Ds^2}{s+2}$$

$$\frac{(s+1)(s+2)(2)-(2s+3)(2s+3)}{\left((s+1)(s+2)\right)^2} = 0 + B + \frac{C(s+1)s^2-(s+1)(2s)}{(s+1)^2} + \frac{D(s+2)s^2-(s+2)(2s)}{(s+2)^2}$$

Now we have isolated B so that once again, we can plug in s = 0 and solve for B.

$$\frac{(1)(2)(2)-(3)(3)}{\left((1)(2)\right)^2} = 0 + B + 0 + 0 \Rightarrow B = -5/4$$

Now we can solve for C and D, once again take equation (i) and multiply by the denominator of the term we are solving for. To solve for C multiply both sides by (s+1)

$$\frac{(2s+3)}{s^2(s+2)} = \frac{A(s+1)}{s^2} + \frac{B(s+1)}{s} + C + \frac{D(s+1)}{s+2}$$

Now with s = -1

$$\frac{(1)}{1(1)} = 0 + 0 + C + 0 \Rightarrow C = 1$$

D is found in exactly the same way, D = ¼, Finally we have all the terms to solve for the first term.

$$\frac{(2s+3)}{s^2(s+1)(s+2)} = \frac{3/2}{s^2} + \frac{-5/4}{s} + \frac{1}{s+1} + \frac{1/4}{s+2}$$

Now we have to separate the second fraction using partial fractions. Since there are no repeated terms this is pretty straight forward, and I will skip the detailed steps.

$$\frac{(s^2+2s+1)}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{1/2}{s} + \frac{0}{s+1} + \frac{1/2}{s+2}$$

Notice that in this case the initial conditions caused a pole/zero cancellation. The implications of such a cancellation will be discussed later in this course.

$$y(t) = L^{-1}\left\{\frac{3/2}{s^2} + \frac{-5/4}{s} + \frac{1}{s+1} + \frac{1/4}{s+2}\right\} + L^{-1}\left\{\frac{1/2}{s} + \frac{1/2}{s+2}\right\}$$

$$y(t) = 3/2t - 5/4 + e^{-t} + 1/4e^{-2t} + 1/2 + 1/2e^{-2t}$$

So the complete solution is given by:

$$\boxed{y(t) = 3/2t - 3/4 + e^{-t} + 3/4e^{-2t}}$$

b)
Given:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = -3\frac{du}{dt} + u$$

$$y(0) = \dot{y}(0) = 0$$

$$u = \begin{cases} t^2 & t > 0 \\ 0 & t \le 0 \end{cases}$$

For this problem I will skip a few more steps than in the first problem. Since we have zero initial conditions for the system the transfer function is simply given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{-3s+1}{s^2+2s+10}$$

Again to find the transfer function we take the inverse Laplace transform. We know that input is t^2 for t > 0. So the its Laplace transform is $\frac{2}{s^3}$

$$y(t) = L^{-1}\{G(s)U(s)\} = L^{-1}\left\{\frac{-3s+1}{s^2+2s+10}\frac{2}{s^3}\right\}$$

To find the inverse Laplace transform we must split the denominator through partial fractions. We have a thrice repeated root s, and a complex denominator. So we will split it up like this:

$$\frac{-6s+2}{s^3(s^2+2s+10)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds+E}{s^2+2s+10}$$

The partial fractions can be solved in the same way as those in part a). Namely multiply first by s^3; solve for A; differentiate; solve for B, differentiate; solve for C. Then solving for D and E.

Here I will illustrate another method for solving partial differential equations.

First multiply both sides by the denominator

$$(-6s+2) = A(s^2+2s+10) + Bs(s^2+2s+10) + Cs^2(s^2+2s+10) + (Ds+E)s^3$$

Since this equation should be true for all s, we can plug in different values of s, obtain a set of simultaneous equations and then solve for the constants A – E.

s = 0	2 = 10 A
s = 1	$-4 = 13A + 13B + 13C + D + E$
s = -1	$8 = 9A - 9B + 9C + D - E$
s = 2	$-10 = 18A + 36B + 72C + 16D + 8E$
s = -2	$14 = 10A - 20B + 40C + 16D - 8E$

Then we can form a matrix :

$$\begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 13 & 13 & 13 & 1 & 1 \\ 9 & -9 & 9 & 1 & -1 \\ 18 & 36 & 72 & 16 & 8 \\ 10 & -20 & 40 & 16 & -8 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} 2 \\ -4 \\ 8 \\ -10 \\ 16 \end{Bmatrix}$$

Solving for this matrix we get :

$$\begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} 1/5 \\ -16/25 \\ 27/250 \\ -27/250 \\ 53/125 \end{Bmatrix}$$

$$L^{-1}\left\{\frac{1/5}{s^3} + \frac{-16/25}{s^2} + \frac{27/250}{s} + \frac{(-27/250)s + 53/125}{s^2+2s+10}\right\}$$

The first 3 terms have straight forward inverse Laplace transforms, but the last term, requires a little algebraic manipulation in order to make it match one of our known forms.

$$\frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}=\frac{A(s+1)+B(3)}{(s+1)^2+3^2}=\frac{-\frac{27}{250}(s+1)+\frac{133}{750}(3)}{(s+1)^2+3^2}$$

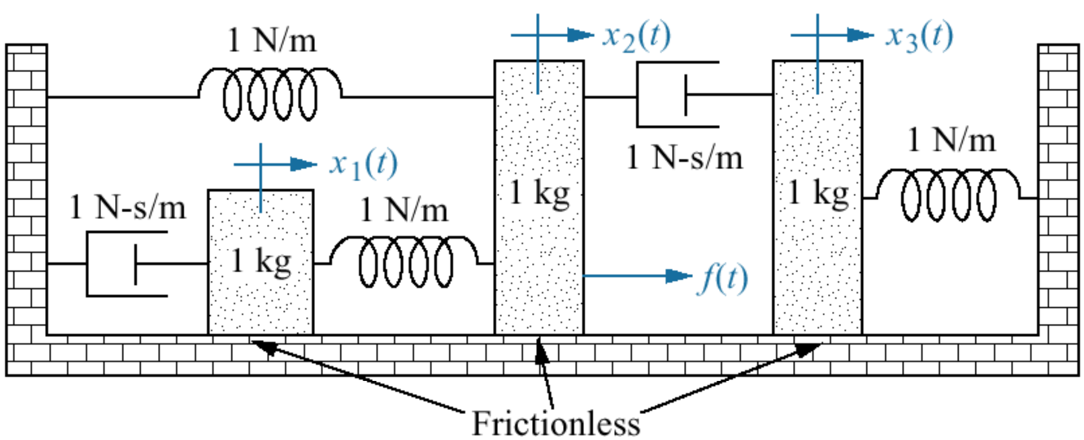
Finally we can take the inverse Laplace transform:

$$\boxed{y(t)=\frac{1}{10}t^2-\frac{16}{25}t+\frac{27}{250}-\frac{27}{250}e^{-t}\cos 3t+\frac{133}{750}e^{-t}\sin 3t}$$

Problem 2

Find the Transfer function $G(s)=X_3(s)/F(s)$

Some of you had problems deriving the equations of motion. This is something that you should already know so I am not directly including it in the solutions. BUT, if you are still having trouble with it…Please read the book. **Ex. 2.17** in the text describes the derivation very clearly. It will take 8 minutes to read.



First, we need to find the equations of motion for this problem. The first step is to notice that we are going to need three equations since the system has three degrees of freedom. Check this by seeing that each mass can move independently of the other two.
Doing a balance of forces, we get:

Mass 1: $$m\ddot{x}_1(t)+f_b\dot{x}_1(t)+kx_1(t)-kx_2(t)=0$$

Which when converted to impedances becomes:

$$(ms^2 + f_b s + k)X_1(s) - dX_2(s) = 0$$

For mass 2 and 3 we will skip directly to the impedances equations:

Mass 2: $$(ms^2 + f_b s + k + k)X_2(s) - kX_1(s) - f_b sX_3(s) = F(s)$$

$$(ms^2 + f_b s + k + k)X_3(s) - f_b sX_2(s) = 0$$

Now we can plug in for m = 1 kg, k = 1 N/m, and f_b = 1 N-s/m.
Notice that we have 3 equations in terms of $X_1(s)$, $X_2(s)$, $X_3(s)$ and F(s).

$$\begin{bmatrix} (s^2+s+1) & -1 & 0 \\ -1 & (s^2+s+2) & -s \\ 0 & -s & (s^2+s+1) \end{bmatrix} \begin{Bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{Bmatrix} = \begin{Bmatrix} 0 \\ F(s) \\ 0 \end{Bmatrix}$$

We want the transfer function to relate $X_3(s)$ to F(s), which can be done either by solving the system of equations through back substitution or through Cramer's rule.
Remember Cramer's Rule:

$$x_k = \frac{\det A_k}{\det A}$$

So for our problem:

$$X_3(s) = \frac{\begin{vmatrix} (s^2+s+1) & -1 & 0 \\ -1 & (s^2+s+2) & F(s) \\ 0 & -s & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -1 & 0 \\ -1 & (s^2+s+2) & -s \\ 0 & -s & (s^2+s+1) \end{vmatrix}} = \frac{s(s^2+s+1)F(s)}{s^6+3s^5+6s^4+8s^3+7s^2+4s+1}$$

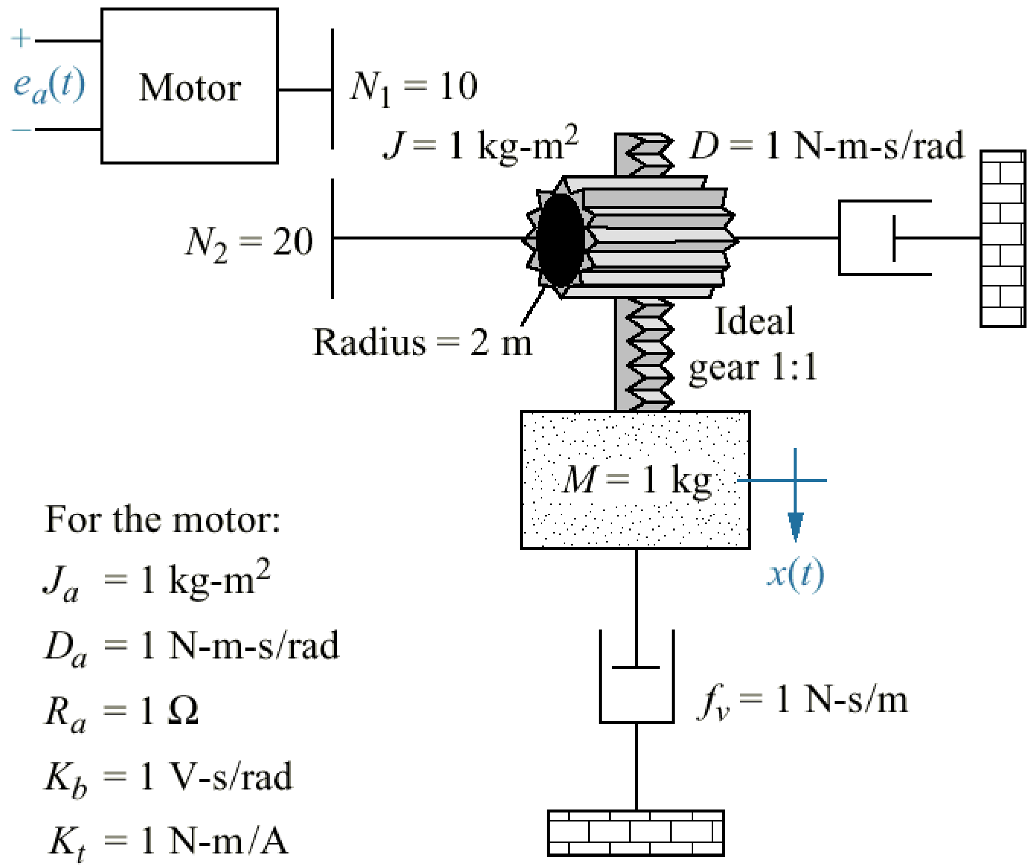
Notice that the denominator can be factored so the final answer is:

$$\boxed{G(s) = \frac{X_3(s)}{F(s)} = \frac{sF(s)}{(s+1)(s^3+s^2+2s+1)}}$$

Which is the same answer you would have gotten through substitution.

Problem 3

Find the transfer function for $\frac{X(s)}{E_a(s)}$.



This problem should not be very complicated if you looked carefully at the book. Section 2.7 goes step by step in the derivation of the transfer function for a DC motor.

$$V_b(s) = K_b s\theta_m(s)$$
$$R_a I_a(s) + V_b(s) = E_a(s)$$
$$I_a(s) = \frac{1}{K_t} T_m(s)$$
$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$$

The above equations combine to form the transfer function for a DC motor as shown below. This derivation is done step by step in the book p. 88 – 90. Please take a look at it if you are confused.

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

Looking at this transfer function we can see that the only variables we don't know are J_m and D_m. In addition we also need to find the relationship between x(t) and $\theta_l(t)$, or:

$$X(s) = r\theta_l(s)$$

To make this problem simpler we do a reflection of the inertia at the motor to combine it with the inertia at the load then we will add the contribution of the mass and reflect the combined inertia back to the motor.

$$T_l(s) = (J_{eq}s^2 + D_m s)\theta_l(s)$$

$$F(s) = (Ms^2 + f_v s)X(s)$$

$$J_{eq} = J_l + \left(\frac{N_2}{N_1}\right)^2 J_a$$

$$D_{eq} = D_l + \left(\frac{N_2}{N_1}\right)^2 D_a$$

$$T_l(s) = \left[\left(J_l + \left(\frac{N_2}{N_1}\right)^2 J_a\right)s^2 + \left(D_l + \left(\frac{N_2}{N_1}\right)^2 D_a\right)s\right]\theta_l(s) + (Ms^2 + f_v s)r^2\theta_l(s)$$

Now that we have all the inertias combined at the side of the load, we can reflect them back to the side of the motor by doing a final transformation across the gears

$$J_m = \left(J_l + \left(\frac{N_2}{N_1}\right)^2 J_a + r^2 M\right)\left(\frac{N_1}{N_2}\right)^2 = 9/4$$

Because of the simplicity of the numbers we get $D_m = 9/4$ as well.

$$\frac{\theta_m(s)}{E_a(s)} = \frac{4/9}{s\left[s + \frac{4}{9}\left(\frac{9}{4} + 1\right)\right]} = \frac{4}{9s^2 + 13s}$$

This is what the question asks for!!! ***Most*** of you did not express the solutions in terms of X(s)/E(s). PLEASE read the question carefully!

This equation must be converted to X(s)/E(s) since:

$X(s) = r\theta_l(s)$ and $2\theta_l(s) = \theta_m(s)$ then $X(s) = \theta_m(s)$

$$\boxed{\frac{X(s)}{E_a(s)} = \frac{4}{9s^2 + 13s}}$$