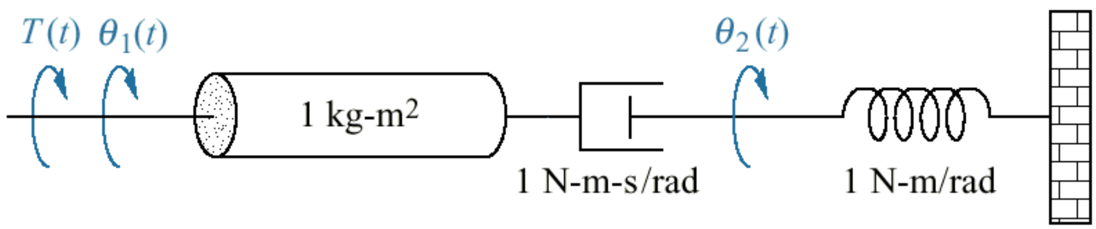
Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics and Control III
Spring 2002

SOLUTIONS: Problem Set #3

Problem 1
Nise 4 – 26



a) Find the Transfer function $G(s) = \theta_2(s)/T(s)$.

First we need to find the equations of motion. Notice that the rotational spring and the inertia can move at different angles. Therefore we will need two equations to get our Transfer function equation. (If you have trouble picturing this, it might help to change it a translational system).

$$T(s) = Js^2\theta_1(s) + Ds\theta_1(s) - Ds\theta_2(s)$$
$$0 = Ds\theta_2(s) - Ds\theta_1(s) + K\theta_2(s)$$

Substituting for the parameters shown, and rearranging the equations we get:

$$T(s) = (s^2 + s)\theta_1(s) - s\theta_2(s)$$
$$\theta_1(s) = \frac{(s+1)}{s}\theta_2(s)$$

Now we can substitute the second equation into the third and we get.

$$T(s) = \cancel{s}(s+1)\left[\frac{(s+1)}{\cancel{s}}\theta_2(s)\right] - s\theta_2(s)$$

$$G(s) = \frac{\theta_2(s)}{T(s)} = \frac{1}{(s^2 + s + 1)}$$

b) The percent overshoot, setting time and peak time for $\theta_2(t)$.

In order to find any of these parameters it is first necessary to find ζ and ω_n. Which are the two 'physically meaningful' parameters necessary to describe a second order system. By knowing these two values you should be able to fully understand the behavior of the system.

The general second order system looks like:

$$G(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Notice that this system has a total gain of 1.

Comparing our system to the general second order system we can find that:

$$\omega_n = 1$$

$$\zeta = 0.5$$

Since the system is a second order system, now we can easily find the values asked for. The following equations are presented in the book along with the derivation.

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 16.3\% \qquad \text{Eq (4.38)}$$

$$T_s = \frac{4}{\zeta\omega_n} = 8 \qquad \text{Eq (4.42)}$$

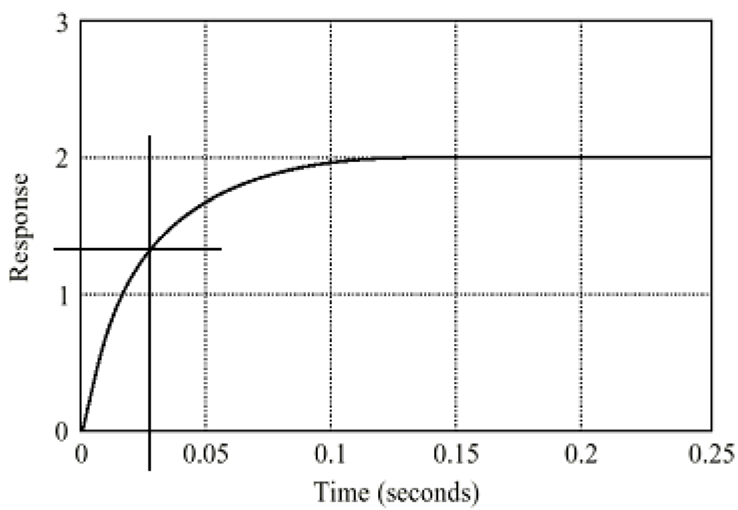
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 3.63 \qquad \text{Eq (4.34)}$$

Problem 2

For each step responses find the transfer function.

a) The most important thing to note here is this is a FIRST ORDER SYSTEM **NOT** and overdamped second order system. The major clue for this is that the initial slope of the response is non-zero. Compare them to b) and c) which both have zero initial slope.

Now since we assume this system is first order. We know that we know that there is only one parameter we want to estimate, namely the time response, τ.



(*a*)

There are several relationships that are defined by the time constant. τ = time it takes for the response to reach 63% of the final value, 4τ = settling time (time it takes for the response to stay within 2% of the final value, or 2.2τ = rise time (time it takes to go from 0.1 to 0.9 of final value. Any of these can be used to deduce the time constant. Choose the one that you think is easier to measure.

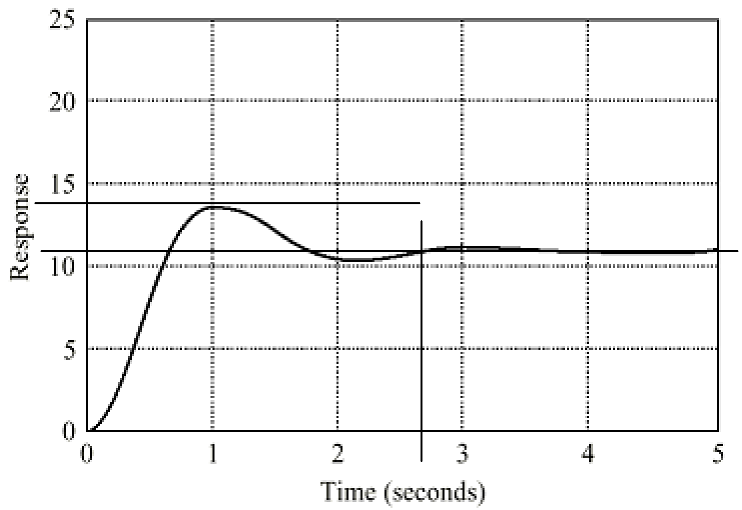
For this case I chose the time it takes to reach 63% of final value (which would be = 1.26), and I approximated this value to be $\tau \cong 0.025$. We know that the general first order system for a system looks like this:

$$G(s) = \frac{K}{s+a}\text{, where a} = 1/\tau.$$

We want know that a = 40, and K/a = 2 since this is the final value of the response. Therefore the transfer function is:

$$G(s) = \frac{80}{s+40}$$

b) Now, this is a second order UNDERDAMPED system, which you should be able to tell by seeing the decaying oscillations. If you are confused about this look a the pictures in p. 190 of Nise.



(b)

For a second order system we need to find ζ and ω_n in order to find the transfer function of the system. Since we know that the percent overshoot is a function of only ζ, we can find the %OS and work backward to get ζ.
First we estimate the percent overshoot to be:

$$\%OS = \frac{c_{\max} - c_{final}}{c_{final}} = \frac{13.5 - 11}{11} \times 100 = 22.7\%$$

Then we can easily find ζ:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.42 \qquad \text{Eq (4.39)}$$

Now that we know ζ we need to find ω_n. Peak time and settling time both relate these two values, so we choose whichever is easiest to get from the graph.
So we estimate the settling time to be, $T_s = 2.8$ s. Therefore.

$$T_s = \frac{4}{\zeta\omega_n} \Rightarrow \omega_n = \frac{4}{\zeta T_s} = 3.4 \qquad \text{Eq (4.42)}$$

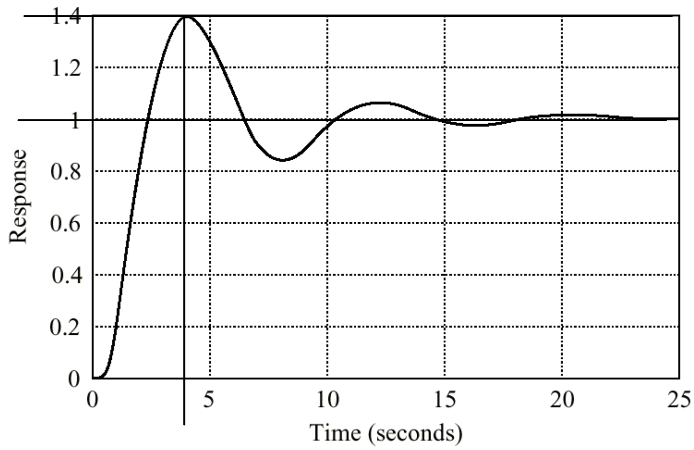
Now we know ζ and ω_n, and can plug these values into the general second order equation,

$$G(s) = \frac{K\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

since this response does not have unity gain, we need to find a value for K that will give us a final value of 11. Namely: $\frac{K\omega_n^2}{\omega_n^2} = 11$, so K = 11 .

$$G(s) = \frac{127.16}{s^2 + 2.86s + 11.56}$$

c) Here we once again, have an underdamped second order system. So I will skip the steps that were described in detail in part b.



(c)

Again we calculate ζ from the %OS. %OS = 40%, so ζ = 0.28.

Now we will use peak time to calculate ω_n.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \omega_n = 0.818$$

Since we do have a final value of 1, the transfer function is:

$$G(s) = \frac{0.669}{s^2 + 0.458s + 0.669}$$

Problem 3

Heading Angle Problem

a) Step Response

Given Transfer function is:

$$G(s) = \frac{\Omega(s)}{\theta_r(s)} = \frac{K(-s+b)}{s+a}$$

where, K = 0.001, a = 0.001, b = 1.

Step Response is given by:

$$\Omega(t) = L^{-1}\left\{\frac{0.001(-s+1)}{(s+0.001}\frac{1}{s}\right\}$$

Separating the fraction through partial fractions we get

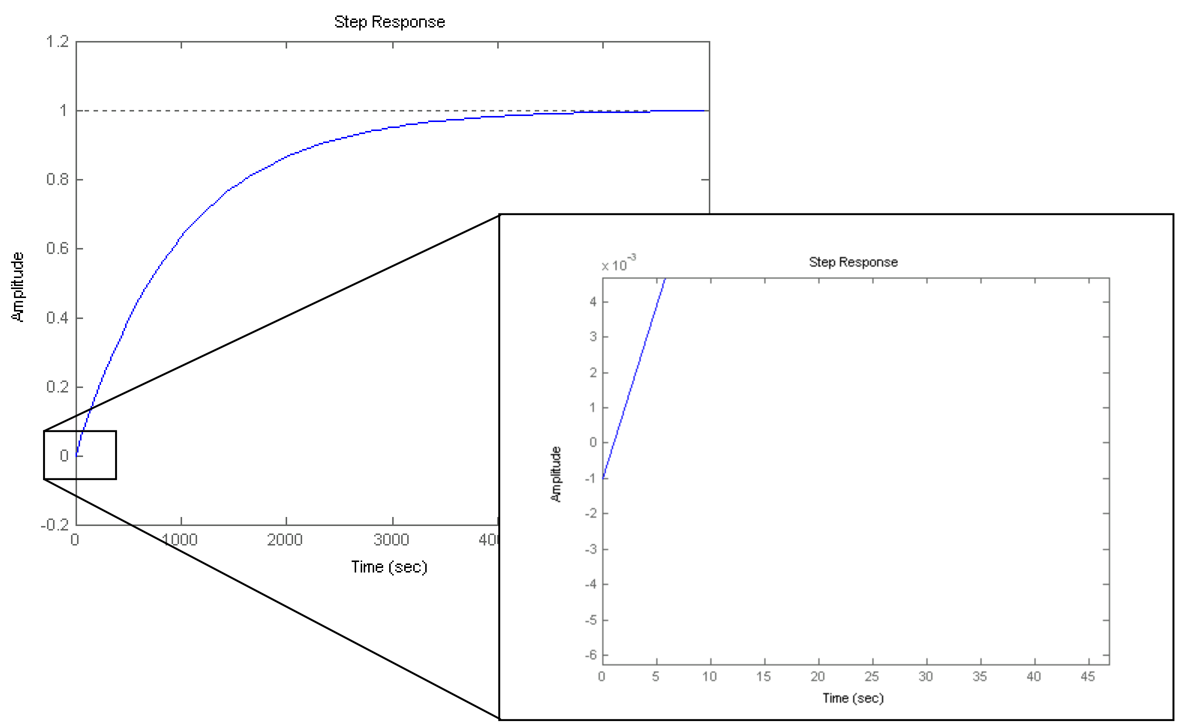
$$\frac{0.001(-s+1)}{(s+0.001}\frac{1}{s} = \frac{A}{s+0.001} + \frac{B}{s} = \frac{-1.001}{s+0.001} + \frac{1}{s}$$

Now taking the inverse Laplace transform we get:

$$\Omega(t) = 1 - 1.001e^{-0.001t}$$

If you have any questions on the step response or partial fractions please look at the solutions for PS#2 again.

Note that at t = 0, the angle rate is negative.



b) Heading angle

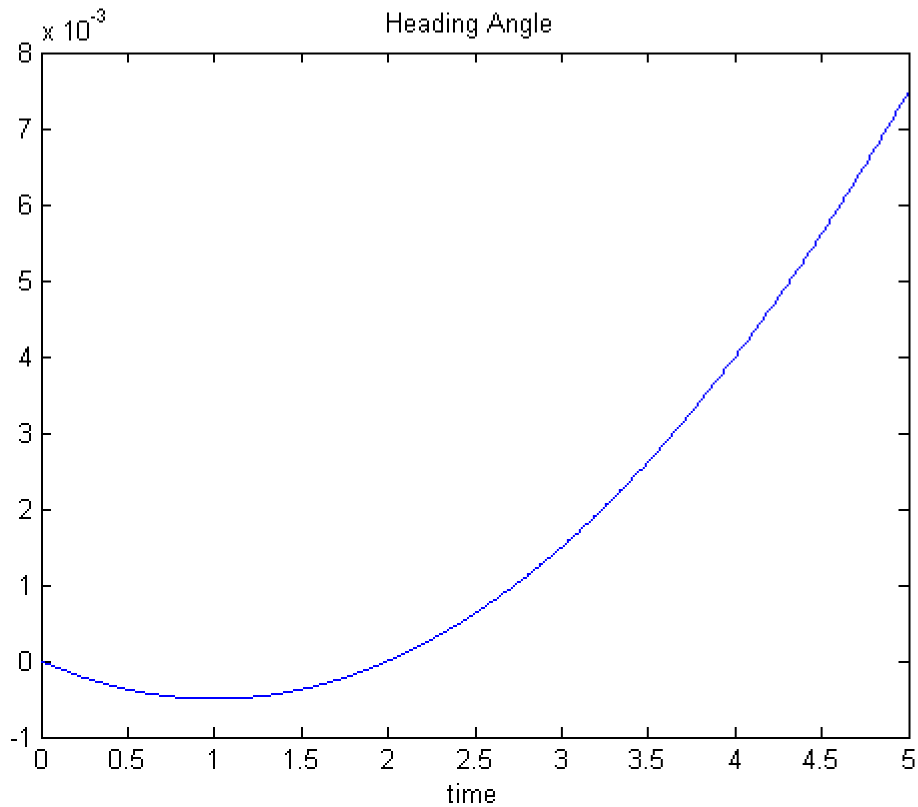
The heading angle as a function of time is given by the integration of the heading angle rate.

$$\theta_H = \int [1 - 1.001e^{-0.001t}]dt$$
$$\theta_H = t + 1001e^{-0.001t} + C$$

since The heading angle is zero initially, $\theta_H(0) = 0$. We find that C = -1001. So,

$$\theta_H = t + 1001e^{-0.001t} - 1001$$

The following plot shows the Heading angle as a function of time.



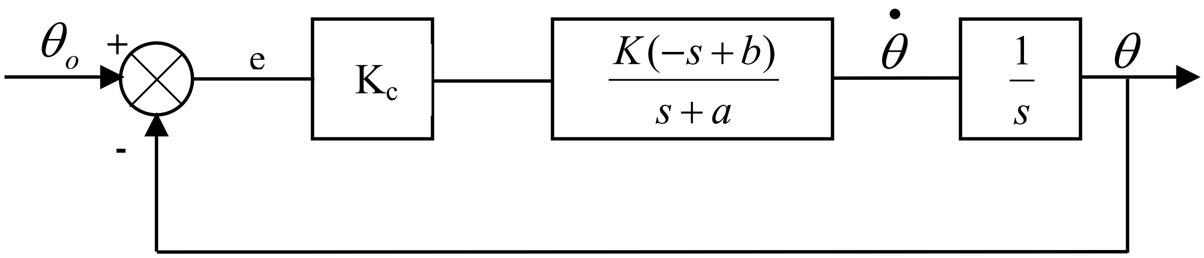
c) Notice that initially the ship is going to be going at an angle in the opposite direction of the specified angle. This is what makes the response non-intuitive. Imagine you want to go right, the ship will begin to turn left initially and *then* turn right, after a few seconds.

d) How long to reach a constant turning rate.

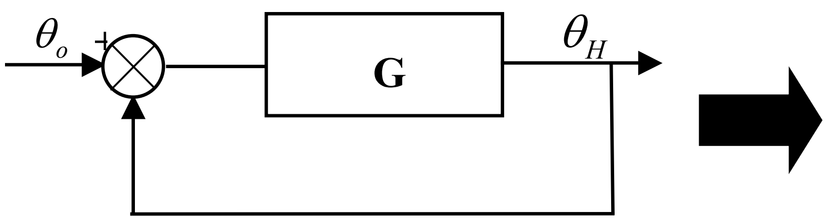
Steady State achieved in 4τ, where τ is 1000, so constant rate after 4000 seconds.

e) Block Diagram

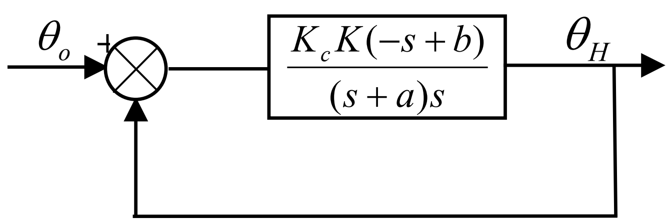
Notice that since our transfer function



The characteristic equation for the closed loop transfer function is given as,



$$T(s) = \frac{G}{1+G}$$



so the characteristic equation is given as,

$$\boxed{\frac{\theta_H}{\theta_o} = \frac{K_c K(-s+b)}{s^2 + (a - K_c K)s + bK_c K}}$$

f.) Value for K → to give an acceptable response…

We want the middle term to be positive in order to have a stable system. Therefore,

$$\left(a - K_c K\right) > 0$$

$$\left(0.001 - K_c \times 0.001\right) > 0$$

so,

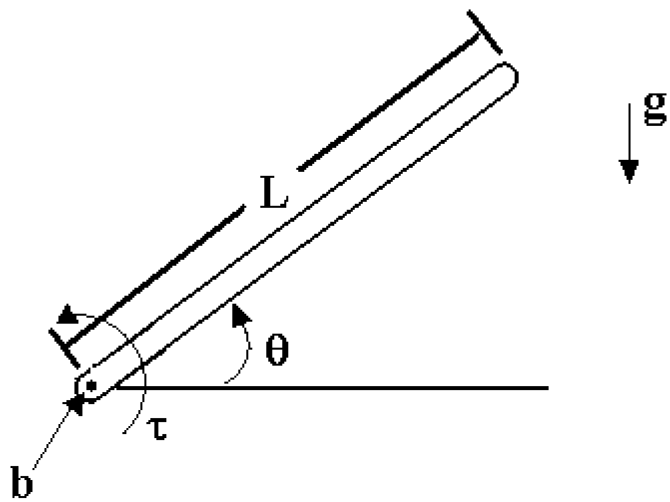
$$\therefore \left(1 - K_c\right) > 0$$

Since K_c must be positive, we need to have a gain:

$$0 < K_c < 1$$

NOTE: The problem set shows the torque τ in the opposite direction, but I sent out an email to have τ and θ in the same direction. However, I solved it here in the correct configuration.

Problem 4



a.) Full non-linear equation:
Doing a torque balance around the pivot point we obtain the following

$$J\ddot{\theta} + B\dot{\theta} + mg\frac{L}{2}\cos\theta = \tau$$

b.) Linear equations for the 3 conditions listed:

The equation above is non-linear because of the cosine term. So we will linearize this term of the equation.

i.) ***arm pointing vertically down***,

$$\theta = -90^o$$

Using the following equation we can get the linear form for cos (θ) when θ = -90.
:

$$f(x) - f(x_0) = \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0)$$

Following the example in the book we let $\theta = \delta\theta - 90$, then we substitute into the equation above. Where f(x) = cos (δθ - 90), f (x_0) = cos (-90), (x – x_0) = δθ

$$\cos(\delta\theta - 90) = \cos(-90) + \left.\frac{d\cos\theta}{d\theta}\right|_{\theta=-90}\delta\theta$$

$$\cos(\delta\theta - 90) = \delta\theta$$

Therefore the full linear equation for this position is:

$$J\ddot{\theta} + B\dot{\theta} + mg\frac{L}{2}\theta = \tau$$

ii.) ***arm pointing vertically up,***

This linearization can be done exactly as that done in part ii), so I will skip the details. So for,

$$\theta = 90^o$$

Note that this term should now have a minus sign

The linearized form is:

$$J\ddot{\theta} + B\dot{\theta} - mg\frac{L}{2}\theta = \tau$$

iii.) ***arm horizontal,***

$$\theta = 0^o$$

Now the linearization of cos θ around θ = 0, is just 1 so for the arm downward this is the linearized equation of motion:

$$J\ddot{\theta} + B\dot{\theta} = \tau - mg\frac{L}{2}$$

Notice that there is an additional torque mgL/2 which is acting on the system.

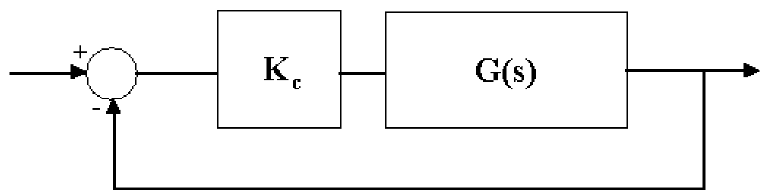
c.) Comparing the 3 motions:

We have not done a rigorous study on stability yet. But you should be able to physically understand what is happening here, by looking at the equation of motion and the transfer function. So, for small motions about the position specified that system will have the following motions:

i.) – stable oscillations
ii.) – unstable oscillations
iii.) – behaves like a mass-damper system with a torque from gravity

d.) Controller

A simple proportional control is applied to the system as shown:



In order to analyze the behavior we must get the characteristic equation, for each possible position. Which is given by:

$$T(s) = \frac{K_c G(s)}{1 + K_c G(s)}$$

i.) For the first system with the following G(s):

$$G(s) = \frac{\theta(s)}{\tau(s)} = \frac{1}{Js^2 + Bs + mg\frac{L}{2}}$$

Characteristic equation for will be given by

$$T(s) = \frac{1}{Js^2 + Bs + \left(mg\frac{L}{2} + K_{c,1}\right)}$$

So,

$$\omega_n^2 = mg\frac{L}{2} + K_{c,1}$$

ii.) For the second system:

$$G(s) = \frac{1}{Js^2 + Bs - mg\frac{L}{2}}$$

where the characteristic equation is

$$T(s) = \frac{1}{Js^2 + Bs + \left(K_{c,2} - mg\frac{L}{2}\right)}$$

and, again the natural frequency is:

$$\omega_n^2 = K_{c,2} - mg\frac{L}{2}$$

iii.) For the last system

$$G(s) = \frac{1}{Js^2 + Bs}$$

Now, the characteristic equation is

$$T(s) = \frac{1}{Js^2 + Bs + K_{c,3}}$$

And the natural frequency is

$$\omega_n^2 = K_{c,3}$$

Now, we are asked to find the values of K_c for which all the natural frequencies are the same.

$$mg\frac{L}{2} + K_{c,1} = -mg\frac{L}{2} + K_{c,2} = K_{c,3}$$

For this to be true K_c must have the following values.

$$K_{c,1} = 0$$

$$K_{c,2} = mgL$$

$$K_{c,3} = mgL/2$$

e.) The gains are different because the effect of gravity at each position is different. Notice that the greatest gain is necessary for the vertical arm. And that for the horizontal arm the gain will equal the torque applied by gravity.

f.) A good control system will have a continuously varying gain as the angle of the bar changes. i.e.: $K_c = f(\theta)$