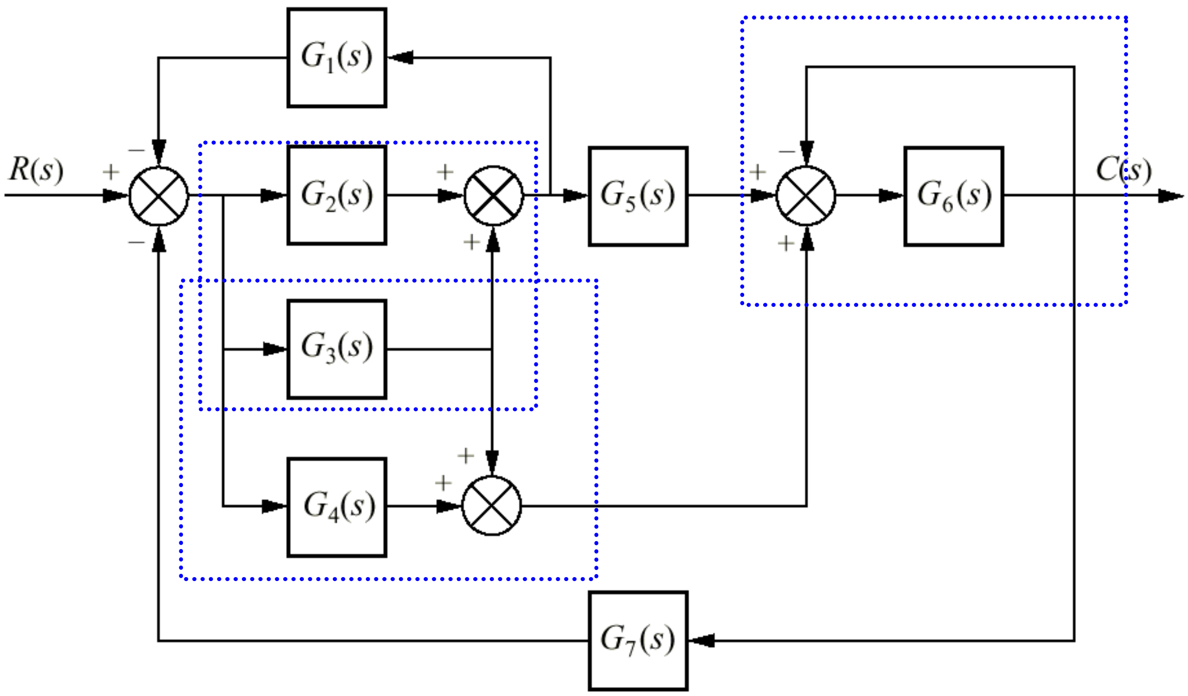
Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics and Control III
Spring 2002

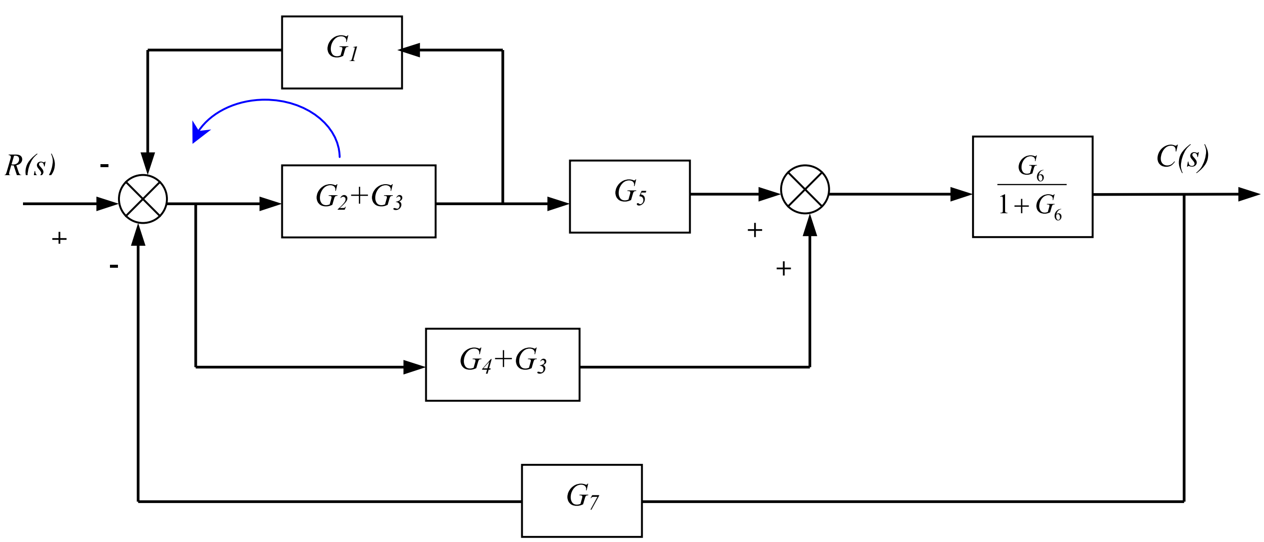
SOLUTIONS: **Problem Set #4**

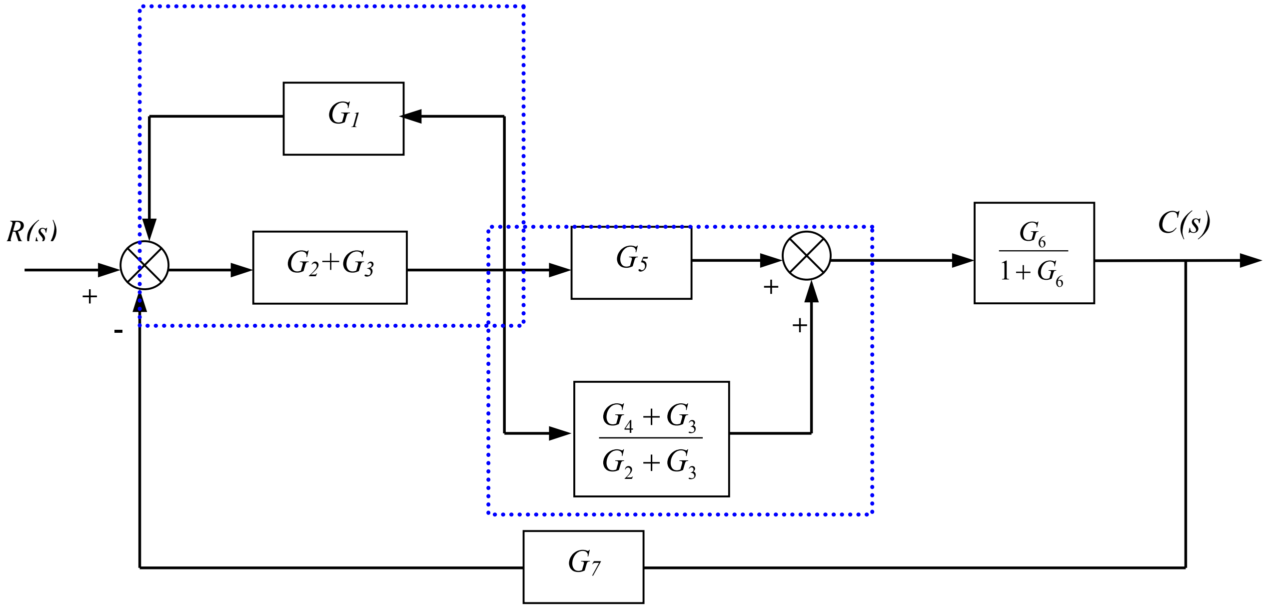
Problem 1



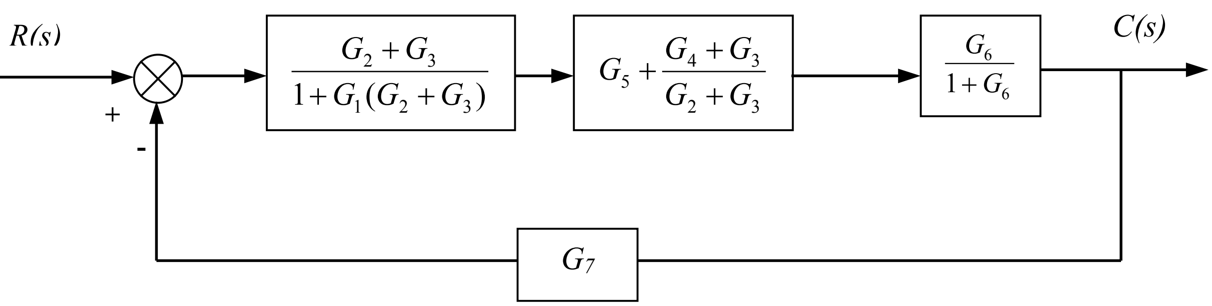
There are several ways of solving the block diagram. As long as you don't break any of the rules any path should lead you to the correct answer.

First Simplify the loop around G_6. Push G_3 through the pick and add it to both G_2 and G_4





Now we can simplify it further by evaluating the two feedback loops.



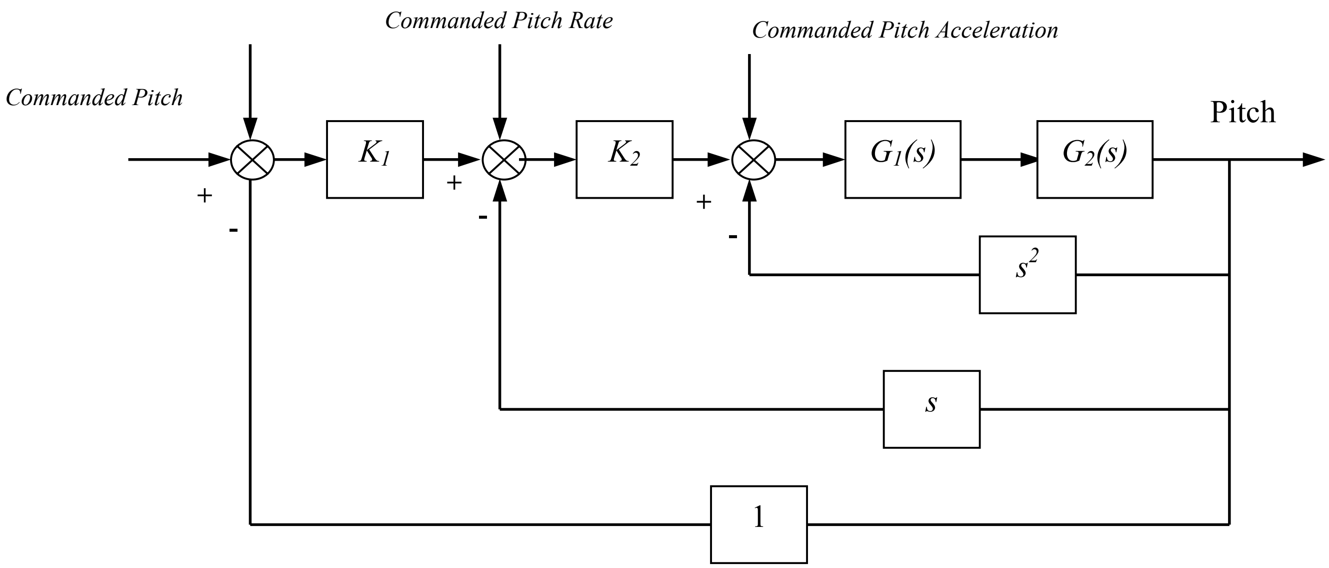
From here it is straight forward just multiply the blocks together and evaluate the feedback loop

$$T(s)=\frac{C(s)}{R(s)}=\frac{\left(\frac{G_2+G_3}{1+G_1(G_2+G_3)}\right)\left(G_5+\frac{G_4+G_3}{G_2+G_3}\right)\left(\frac{G_6}{1+G_6}\right)}{1+G_7\left(\frac{G_2+G_3}{1+G_1(G_2+G_3)}\right)\left(G_5+\frac{G_4+G_3}{G_2+G_3}\right)\left(\frac{G_6}{1+G_6}\right)}$$

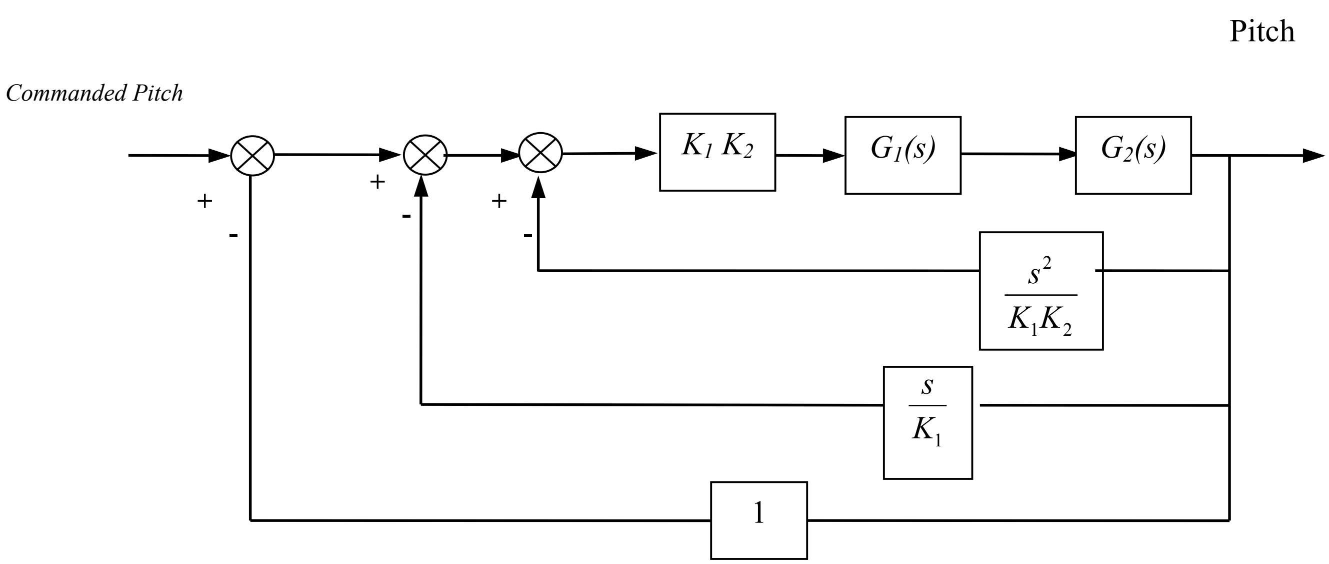
$$T(s)=\frac{G_6(G_5G_2+G_5G_3+G_4+G_3)}{1+G_6+G_1G_2+G_1G_2G_6+G_1G_3+G_1G_3G_6+G_7G_6G_5G_2+G_7G_6G_5G_3+G_7G_6G_4+G_7G_6G_3}$$

Problem 2

a) The first part asks for the relation between the a commanded pitch input and the output being the actual pitch. This should be the simplest transfer



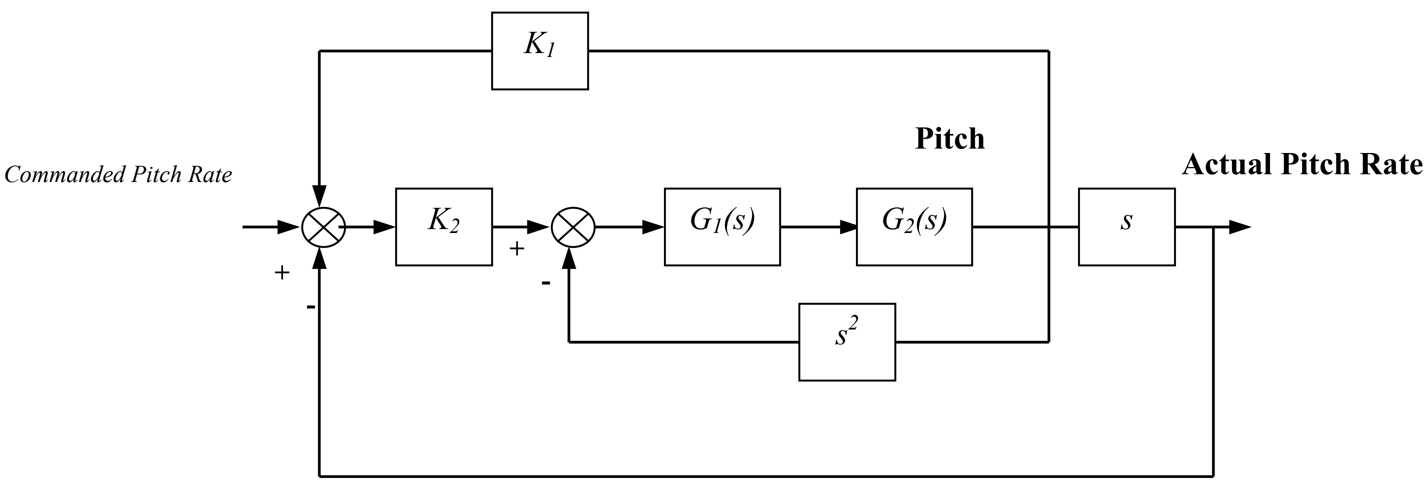
There are several ways to reduce this block diagram, but the simplest is probably by moving the K_1 and K_2 forward, and then taking collapsing the summing junctions.



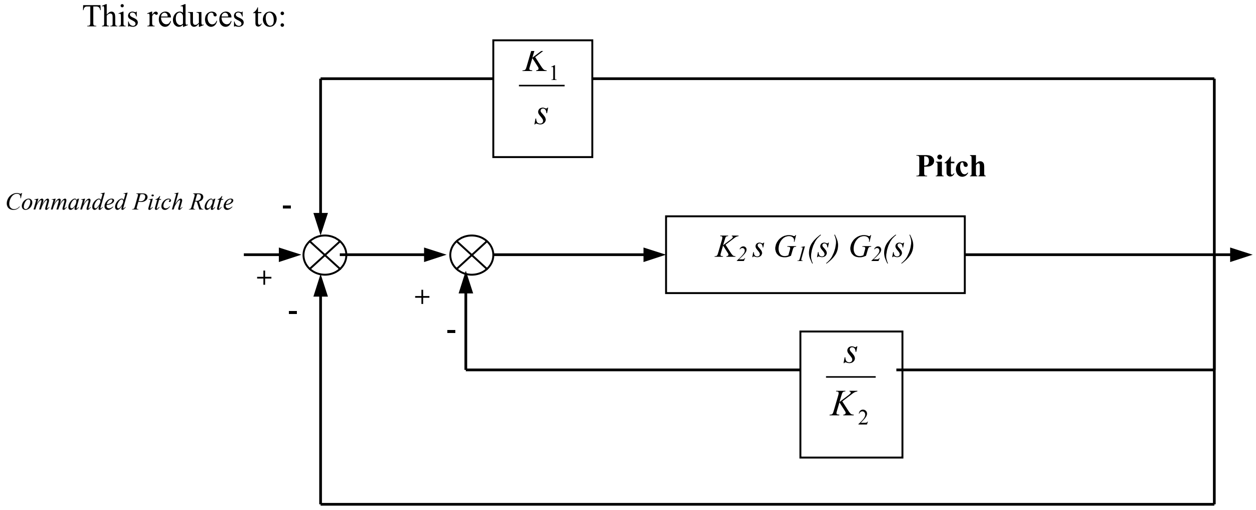
$$T(s) = \frac{K_1K_2G_1(s)G_2(s)}{1 + K_1K_2G_1(s)G_2(s)\left(1 + \frac{s}{K_1} + \frac{s^2}{K_1K_2}\right)}$$

b)

This part asks for the transfer function between the Commanded pitch rate and the actual pitch rate. The first thing to understand is that you CANNOT cut off the parts of the block diagram that come before and after the place where these signals lie. You **must** keep all the existing signals. Here I show the rearranged transfer function so that your input and output are in the places expected, and all the connection signals remain the same.



This reduces to:

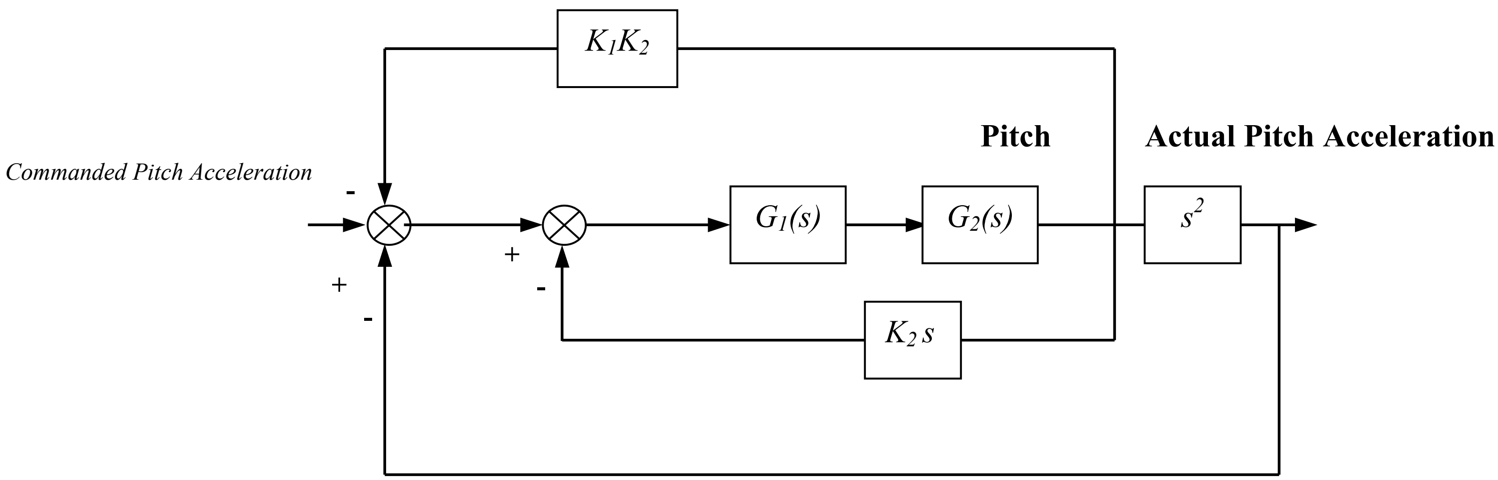


The transfer function is:

$$T(s) = \frac{K_2\, s\, G_1(s) G_2(s)}{1 + K_2\, s\, G_1(s) G_2(s)\left(1 + \frac{s}{K_2} + \frac{K_1}{s}\right)} = \frac{K_2\, s\, G_1(s) G_2(s)}{1 + G_1(s) G_2(s)\left(s^2 + K_2 + K_1 K_2\right)}$$

c)

For this part we need to do essentially the same thing as in part b.
We set all inputs to zero except for the commanded pitch acceleration and find where the actual pitch acceleration is found.



Simplifying the block diagram by moving the s^2 block backwards and collapsing the summing junctions we get the following transfer function:

$$T(s) = \frac{s^2 G_1(s)G_2(s)}{1+s^2 G_1(s)G_2(s)\left(1+\frac{K_1K_2}{s^2}+\frac{K_2}{s}\right)} = \frac{s^2 G_1(s)G_2(s)}{1+G_1(s)G_2(s)\left(s^2+K_2 s+K_1K_2\right)}$$

Problem 3
Nise 4-20 a, b only
This is very straight forward.

$$T(s) = \frac{121}{s^2 + 13.2s + 121}$$

$$\omega_n = 11$$

$$2\zeta\omega_n = 13.2 \Rightarrow \zeta = 0.6$$

$$T_s = \frac{4}{\zeta\omega_n} = 0.606$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.357$$

$$\%OS = e^{\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 9.48\%$$

From the approximation given on the bottom of page 196:

$$\omega_n T_r = 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1 \Rightarrow T_r = 0.168\,\text{sec}$$

b)

$$T(s) = \frac{0.04}{s^2 + 0.02s + 0.04}$$

$$\omega_n = 0.2$$

$$2\zeta\omega_n = 0.02 \Rightarrow \zeta = 0.05$$

$$T_s = \frac{4}{\zeta\omega_n} = 400$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 15.73$$

$$\%OS = e^{\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 85.45\%$$

$$\omega_n T_r = 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1 \Rightarrow T_r = 5.26\,\text{sec}$$

Problem 4

b) This problem is very similar to the ones done in PS #3

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.491$$

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = 7.21$$

Now you have enough information to create the denominator of the transfer function and solve for the poles.
If you want to avoid that extra step, you should know that the poles of a second order system are given by

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$p_{1,2} = -3.54 \pm j6.28$$

c) T_s = 7 sec, T_p = 3 sec.

This problem becomes even simpler by noting that:

$$\zeta\omega_n = \frac{4}{T_s} = 0.571$$

$$\omega_n\sqrt{1-\zeta^2} = \frac{\pi}{T_p} = 1.047$$

Therefore:

$$p_{1,2} = -0.571 \pm j1.047$$

Problem 5

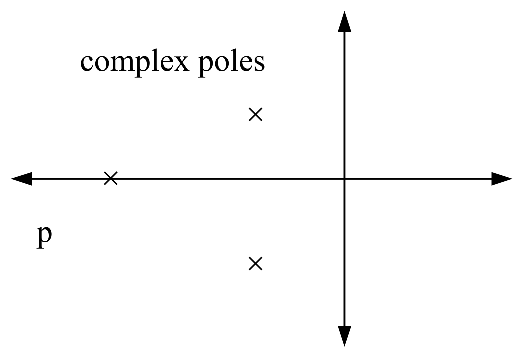
a)

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$y(t) = 1 + Ae^{-pt} + Be^{-\sigma t}\sin(\omega_d t - \theta)$$

Given the above transfer function and step input response we can analyze how the poles behave.

First it might be helpful to plot the poles in a pole-zero map to understand what we are working with.



The dominant poles are closest to the imaginary axis. So as p gets large the complex poles dominate the response. This is also evident because the term Ae^{-pt} would get small, so it's contribution is smaller.

b)
As p gets small p → 0

$$A = \frac{-\omega_n^2}{\omega_n^2 - 2\zeta\omega_n p + p^2}$$

A → -1

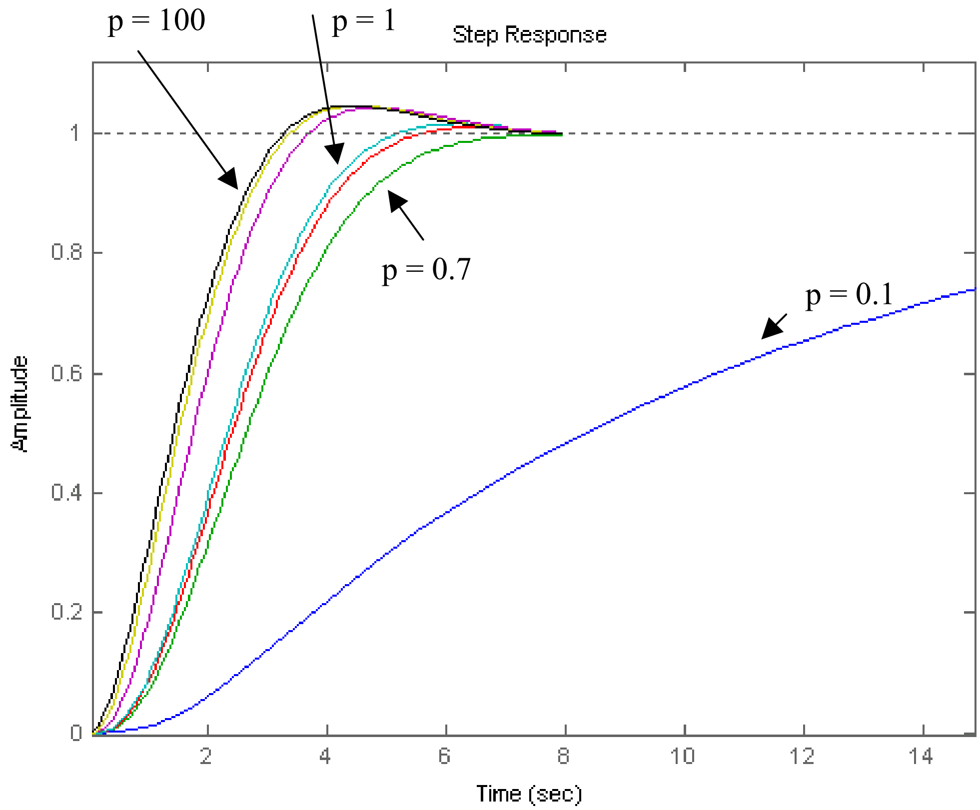
While

$$B = \frac{p}{\sqrt{(p^2 - 2\zeta\omega_n p + \omega_n^2)(1-\zeta^2)}}$$

B → 0

c) This clearly shows that as p gets smaller the it becomes the dominant pole. Smaller p with respect to the distance from the imaginary axis

d) The following plot was obtained using the step function in matlab.



When p gets greater than 0.7 = $\zeta\omega_n$ it moves to the left of the complex poles and the complex poles begin to take over. You can see this because this is where the overshoot first appears.
As p gets larger the response approaches the response for

$$H(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

which when plotted for the values given looks like: