Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics and Control III
Spring 2002

SOLUTIONS: Problem Set #5

Problem 1

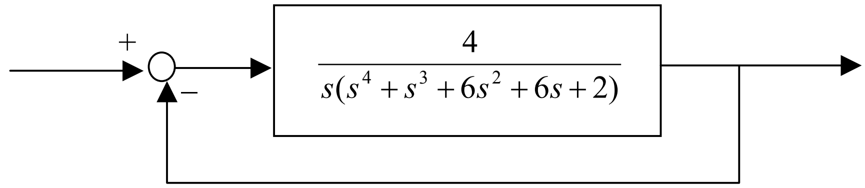
Routh-Hurwitz table for the open loop system shown

$$G(s) = \frac{2s^2 + 3s + 8}{s^4 + 2s^3 + 4s^2 + 6s + 8}$$

s^4	1	4	8
s^3	$\cancel{2}$ 1	$\cancel{6}$ 3	0
s^2	$\dfrac{-\begin{vmatrix}1 & 4\\ 1 & 3\end{vmatrix}}{1} = 1$	8	0
s^1	-5	0	0
s^0	8	0	0

This tells us that there are 2 poles in the RHP

Problem 2



The characteristic equation for the closed loop transfer function given by the block diagram above is:

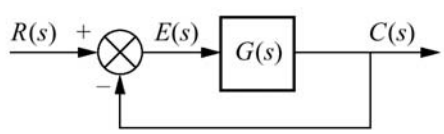
$$s^5 + s^4 + 6s^3 + 6s^2 + 2s + 4$$

s^5	1	6	2
s^4	1	6	4
s^3	$\emptyset \to \varepsilon$	-2	0
s^2	$\frac{2+6\varepsilon}{\varepsilon}$	4	0
s^1	$\frac{-(4\varepsilon^2+4+12\varepsilon)}{2+6\varepsilon}$	0	0
s^0	4	0	0

		$\varepsilon > 0$	$\varepsilon < 0$
s^5	1	+	+
s^4	1	+	+
s^3	ε	+	-
s^2	$\frac{2+6\varepsilon}{\varepsilon}$	+	-
s^1	$\frac{-(4\varepsilon^2+4+12\varepsilon)}{2+6\varepsilon}$	-	+
s^0	4	+	+

Note that whether epsilon is greater than zero or less than zero, you still get two poles on the right hand plane. You should always get the same result for these two conditions.

Problem 3



$$G(s) = \frac{K}{s(s+1)(s+2)(s+5)}$$

Closed loop transfer function:

$$T(s) = \frac{K}{s^4 + 8s^3 + 17s^2 + 10s + K}$$

s^4	1	17	K
s^3	~~8~~ → 4	~~10 → 5~~	0
s^2	~~63/4~~ → 63	~~K~~ → 4K	0
s^1	315 - 16K	0	0
s^0	4K	0	0

For stability we must have:

$$4K > 0 \text{ and } (315 - 16K) > 0$$

solving for both equations we get:

$$0 < K < 315/16$$

b) Marginal stability implies that there is a row of zeros. This occurs for s^1 when

$$K = 315/16$$

c) When the system is marginally stable our transfer function looks like this:

$$T(s) = \frac{315/16}{s^4 + 8s^3 + 17s^2 + 10s + 315/16}$$

To find all the poles of the system at this point we need to factor the denominator. This task becomes significantly simpler since we know that this value of K makes the row $s^1 = 0$.

s^4	1	17	315/4
s^3	4	5	0
s^2	63	315/4	0
s^1	0	0	0
s^0	315	0	0

So, we know that the coefficients in the row s^2 form a polynomial that will be a factor of the original denominator. This polynomial is:

$$63s^2 + 315/4$$

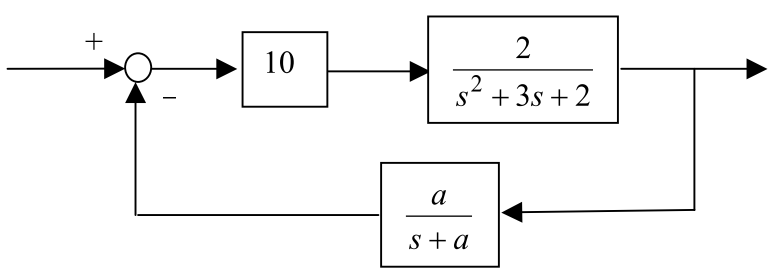
Now we can get the other factor of the equation by dividing the original polynomial. Finally we get:

$$(63s^2 + 315/4)(1/63s^2 + 8/63s + 1/4)$$

So the poles can now be found easily through the quadratic function.

$$p_{1,2} = \pm i\sqrt{5/4}$$
$$p_{3,4} = -4.5, -3.5$$

Problem 4



The first step here is to find the closed loop transfer function for the system, which reduces to:

$$T(s) = \frac{20a}{s^3 + (3+a)s^2 + (2+3a)s + 22a}$$

s^3	1	2 + 3a
s^2	3 + a	22a
s^1	$\frac{3a^2 - 11a + 6}{3+a}$	0
s^0	22a	0

For the system to be stable we must have:

$$3 + a > 0$$

$$\frac{3a^2 - 11a + 6}{3+a} > 0, \text{ and}$$

$$22a > 0$$

So, $a > 0$ from the third line, and $(3a - 2)(a - 3) > 0$

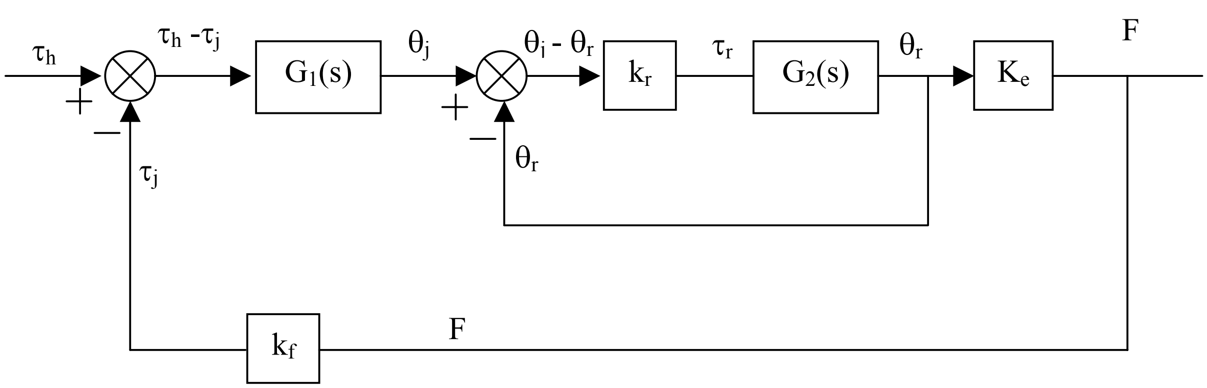
From the second line: $a > 3$, and $a < 2/3$
Together the requirements are:

$$0 < a < 2/3, \text{ and } a > 3$$

Problem 5

There are two major ways of solving this problem. The first (which many of you used) is to get all the equations that are given and those derived from the diagram and solve the system of equations.
Since most of you did it this way, I am going to show it using block diagrams which is a bit cleaner way to do it.

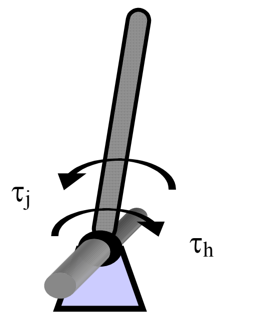
The first thing to do is recreate the block diagram that is described by the picture shown.

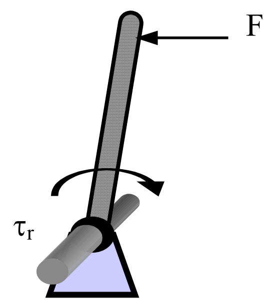


The block diagram as shown has all the connections described by the equations. For instance notice that $\tau_j = -k_j(\theta_j - \theta_r),$ is created around the second summing junction. $G_1(s)$ and $G_2(s)$ represent the characteristic of each arm. For $G_1(s)$ we need to find the transfer function between τ_h - τ_j and θ_j, while for $G_2(s)$ we need to find the transfer function between τ_r and θ_r.

Doing a sum of forces around the rods we get:

$$\sum \tau = J\ddot{\theta}$$





$$\tau_h - \tau_j = J_J \ddot{\theta}_J$$

$$\therefore \; G_1(s) = \frac{1}{J \;\; s^2}$$

$$J_r\ddot{\theta}_r = \tau_r - F(l)$$

$$F = K_e \theta_r l$$

$$\therefore G_2 = \frac{1}{J_r s^2 + K_e}$$

Now we can start simplifying the block diagram to get the response. The easiest way to do this is to find the transfer function $\frac{F}{\tau_h}$ and then find the transfer function $\frac{F}{\theta_j}$ and then divide the first one by the second to get $\frac{\theta_j}{\tau_h}$

To get this transfer function first reduce the inner feedback system and then the outer feedback system.

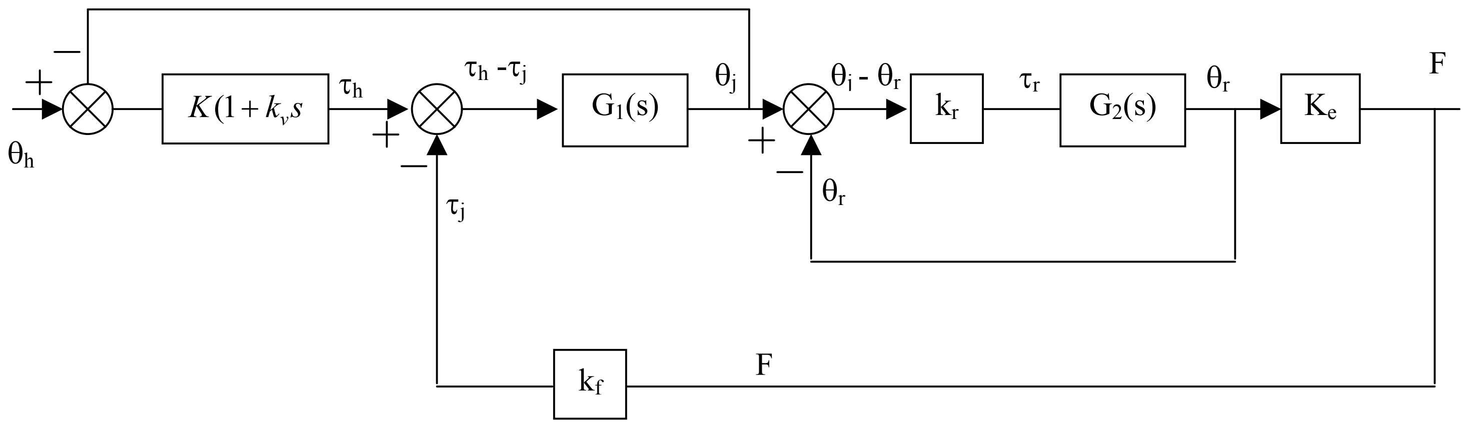
$$\frac{F}{\tau_h} = \frac{\dfrac{k_r K_e}{s^2 J_j (s^2 J_r + K_e + k_r)}}{1 + \dfrac{k_r K_e}{s^2 J_j (s^2 J_r + K_e + k_r)}} = \frac{k_r K_e}{s^2 J_j (s^2 J_r + K_e + k_r) + k_r K_e}$$

To get this transfer function first reduce the inner feedback system and then find the open loop transfer function between the input and output.

$$\frac{F}{\theta_h} = \frac{k_r K_e}{(s^2 J_r + K_e + k_r)}$$

$$\boxed{\frac{\theta_j}{\tau_h} = \frac{s^2 J_r + K_e + k_r}{s^4 J_J J_r + (K_e + k_r) s^2 J_J + k_r K_e k_f}}$$

b) Control: Proportional and derivative control.



Here is the new block diagram including the proportional-derivative control. Now we are looking for the transfer function: $\frac{\theta_j}{\theta_h}$. Again, we could reduce the block diagram to find this transfer function. But for simplicity we should use the equation given:

$\tau_h(s) = K(1+k_v s)[\theta_h(s) - \theta_j(s)]$

which shows the relationship already in terms of θ_h, τ_h, and θ_j so all we need to do is find relationship between τ_h and θ_j. -- which is exactly what we did in part a.

So,

$$\frac{(s^4 J_J J_r + (K_e + k_r)s^2 J_J + k_r K_e k_f)\theta_j(s)}{s^2 J_r + K_e + k_r} = K(1+k_v s)[\theta_h(s) - \theta_j(s)]$$

$$(s^4 J_J J_r + (K_e + k_r)s^2 J_J + k_r K_e k_f)\theta_j(s) = K(s^2 J_r + K_e + k_r)(1+k_v s)[\theta_h(s) - \theta_j(s)]$$

$$\frac{\theta_j(s)}{\theta_h(s)} = \frac{K(s^2 J_r + K_e + k_r)(1+k_v s)}{(s^4 J_J J_r + (K_e + k_r)s^2 J_J + k_r K_e k_f) + K(s^2 J_r + K_e + k_r)(1+k_v s)}$$

Now we look at this denominator to obtain stability conditions.

We simplify the denominator using the following given parameters:

$$J_r = 2,\ J_j = 1,\ k_r = 2,\ K_e = 8,\ k_f = 0.5$$

$$2s^4 + 2Kk_v s^3 + (2K+10)s^2 + 10Kk_v s + (10K+8)$$

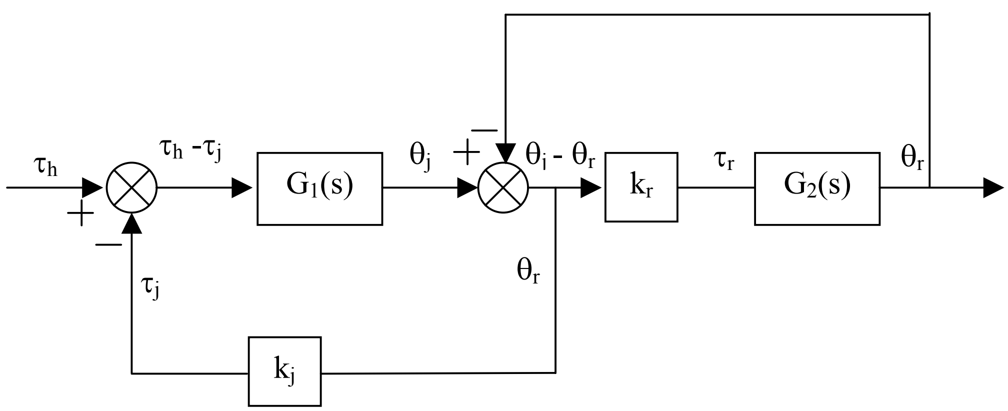
Now we can build the Routh Table:

s^4	$\cancel{2} \rightarrow 1$	$\cancel{(2K+10)} \rightarrow (K+5)$	$\cancel{(10K+8)} \rightarrow (5K+4)$
s^3	$\cancel{2Kk_v} \rightarrow 1$	$\cancel{10Kk_v} \rightarrow 5$	0
s^2	K	5K+4	0
s^1	-4/K	0	0
s^0	5K+4	0	0

Because $K \neq 0, k_v \neq 0$, we can factor out both of these values from the second line. As a result notice that k_v does not influence the stability.

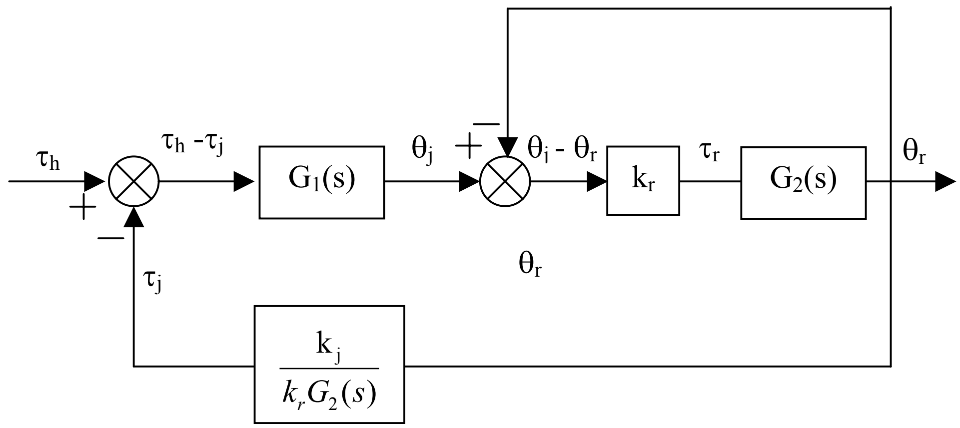
This system is always unstable because s^2 and s^1 have conflicting requirements for K to make their respective lines positive.

c) The modified diagram for the bilateral servo control is shown.



Again, the first thing we do, is to find the transfer function between $\frac{\theta_j}{\tau_h}$, assuming that we have not yet applied the proportional-derivative control. which we will do in exactly the same method used in part a). by first finding the $\frac{\theta_r}{\tau_h}$ transfer function and then finding the $\frac{\theta_r}{\theta_j}$ and then divide the first one by the second to get $\frac{\theta_j}{\tau_h}$

Moving k_r and $G_2(s)$ before the pickoff point makes this problem a lot simpler.



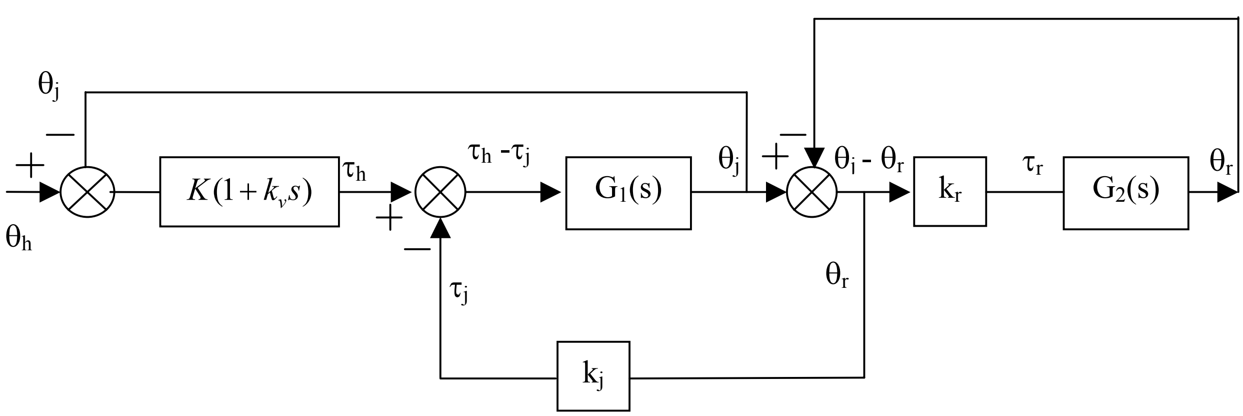
$$\frac{\theta_r}{\theta_j} = \frac{k_r G_2(s)}{1 + k_r G_2(s)}$$

$$\frac{\theta_r}{\tau_h} = \frac{\dfrac{k_r G_2(s)}{1 + k_r G_2(s)} G_1(s)}{1 + \dfrac{k_r G_2(s)}{1 + k_r G_2(s)} G_1(s) \dfrac{\mathrm{k_j}}{k_r G_2(s)}}$$

$$\frac{\theta_r}{\tau_h} = \frac{1/J_j s^2}{1 + \dfrac{1}{1 + k_r(1/J_r s^2 + K_e)} \dfrac{1}{J_j s^2} \dfrac{k_j}{1}}$$

$$\boxed{\frac{\theta_j}{\tau_h} = \frac{J_r s^2 + k_r + K_e}{J_r J_j s^4 + [J_r k_j + J_j(k_r + K_e)]s^2 + k_j K_e}}$$

Now we are ready to apply the PD control:



We follow the same steps to get the new closed loop transfer function

$$\frac{\theta_j(s)}{\theta_h(s)} = \frac{K(s^2 J_r + K_e + k_r)(1 + k_v s)}{(s^4 J_J J_r + [(K_e + k_r)J_J + k_j J_r]s^2 + k_j K_e) + K(s^2 J_r + K_e + k_r)(1 + k_v s)}$$

The characteristic equation is now:

$$2s^4 + 2Kk_v s^3 + (2K + 14)s^2 + 10Kk_v s + (10K + 16)$$

The Routh table is similar to part b)

s^4	$\cancel{2} \rightarrow 1$	$\cancel{(2K+14)} \rightarrow (K+7)$	$\cancel{(10K+16)} \rightarrow (5K+8)$
s^3	$\cancel{2Kk_v} \rightarrow 1$	$\cancel{10Kk_v} \rightarrow 5$	0
s^2	K+2	5K+8	0
s^1	$\frac{2}{K+2}$	0	0
s^0	5K + 8	0	0

Now we can clearly see that for all positive gains $K > 0, k_v > 0$, the system will be stable.