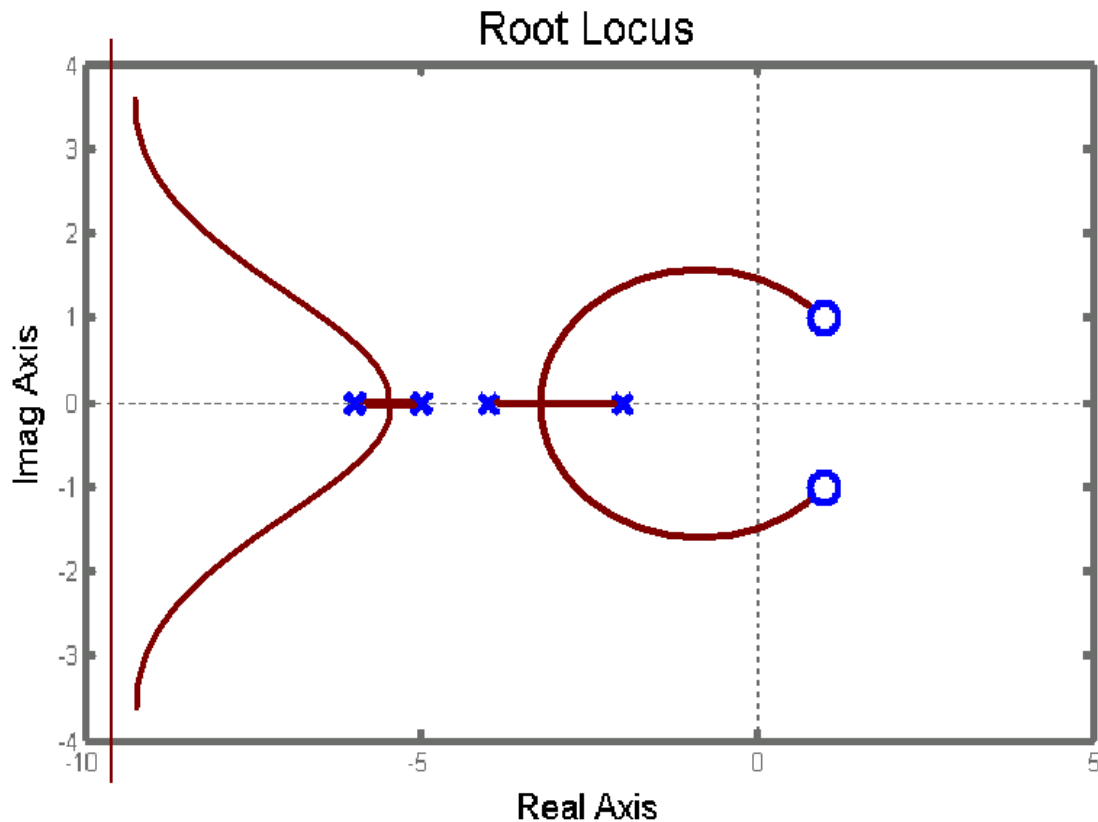**SOLUTIONS**: Problem Set # 8

Problem 1   Nise 8-25

a)   Sketch the Root Locus

Open loop transfer function:

$$G(s) = \frac{K(s^2 - 2s + 2)}{(s+2)(s+4)(s+5)(s+6)}$$

You should be able to sketch the root locus as shown above by hand. You must calculate the location and number of your asymptotes, but you may leave out characteristics that define your sketch such as breakaway/in points and departure/arrival angles.

b)   Asymptotes

Using Professor Asada's notation

$$\alpha = \frac{(z_1 + z_2) - (p_1 + p_2 + p_3 + p_4)}{n - m} = \frac{(-1 + j - 1 - j) - (2 + 4 + 5 + 6)}{2} = -9.5$$

Using Nise's notation:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} = \frac{(-2 - 4 - 5 - 6) - (+1 + j + 1 - j)}{2} = -9.5$$

Notice that these are equivalent. So don't get confused by the notation.

c)   Gain that makes system stable

Using the Routh Hurwitz table:

First obtain the characteristic equation is:
$$C.E. = s^4 + 17s^3 + (104 + K)s^2 + (268 - 2K)s + (240 + 2K)$$

s^4	1	104 + K	240 + 2K
s^3	17	268 - 2K	
s^2	$\dfrac{1500 + 19K}{17}$	240 + 2K	
s^1	$\dfrac{332640 + 1514K - 38K^2}{1500 + 19K}$		
s^0	240 + 2K		

To for the range of stable K we must make a row of zeros.
Solving the quadratic equation of s^1 for K we get:

$$K = 115.6, -75.7$$

We can also get the point where the Root locus crosses the $j\omega$ axis, by building the factor from the line s^2. We use $K = 115.6$, so the equation:

$$217.4s^2 + 471.2 = 0$$

will give the location of the poles at the imaginary axis, and the value of ω at this point is:

$$\omega = \pm j1.47$$

d)   Breakaway points:

There are a few ways of getting the breakaway points. The book shows how to do it using the maximum (breakaway) and minimum (break-in) value for K along the real segment where the root locus breaks.

$$K = \frac{-1}{G(\sigma)} = \frac{s^4 + 17s^3 + 104s^2 + 268s + 240}{s^2 - 2s + 2}$$

$$\frac{dK}{d\sigma} = \frac{(2s-2)(s^4 + 17s^3 + 104s^2 + 268s + 240) - (4s^3 + 51s^2 + 208s + 268)(s^2 - 2s + 2)}{(s^2 - 2s + 2)^2} = 0$$

Solving this equation becomes very impractical and should be done either with a graphing calculator or matlab. Instead at this point it becomes easier to use matlab to plot the root locus and find the breakaway/in points.

Root Locus
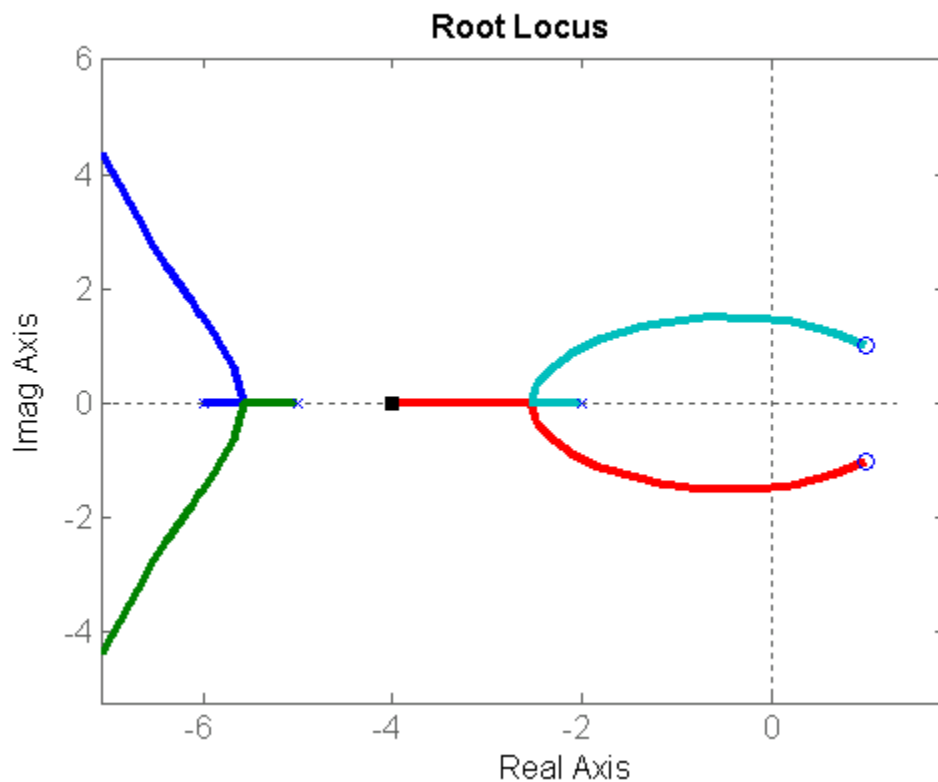
There are a few ways to get the breakaway/in points in matlab as well. The simplest way is zooming in around the area. You should also check out the command rlocfind it is very useful.

The breakaway point between –2 and –4 is –2.53
The breakaway point between –5 and –6 is –5.58
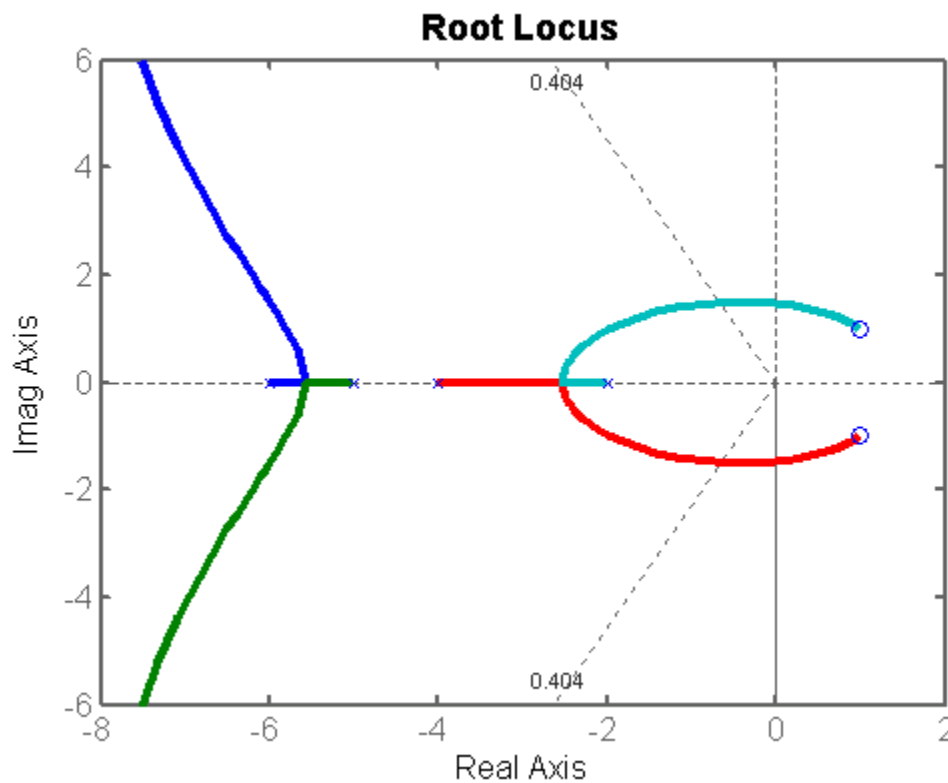
There are no break-in points.

e)   K at 25% Overshoot.

A 25% O.S. yields:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.404$$

Graphically there will be constant %O.S of 25% along the line that goes through the origin with a slope of:

$$m = -\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = -2.26$$

Root Locus

Now you need to find the point where the Root Locus intersects the line for $\zeta = 0.404$.

The best way to do this is by again using the command rlocfind. Using rlocfind will give you the K value as well as the location of all the poles.

$$K = 36.03$$
$$p_{1,2} = -0.66 \pm 1.50i$$
$$p_{3,4} = -7.84 \pm 7.43i$$

f)   Higher order poles

The poles $p_{1,2}$ can be used for the second order approximations. While poles $p_{3,4}$ are the higher order poles when the system operates at 25% O.S.

g)   This second order approximation is not valid because there are two zeros on the right hand plane that will change the response.
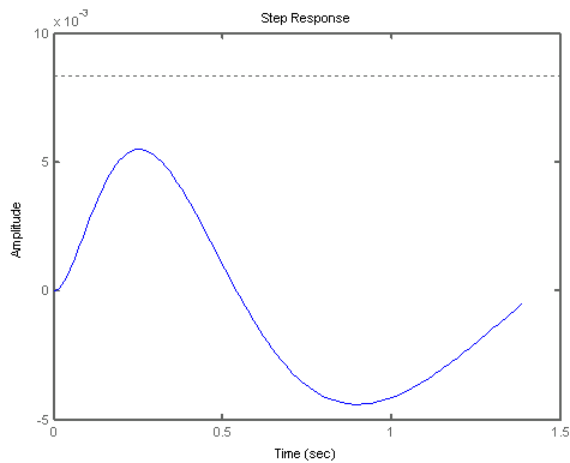
h)   Step response.

The step response given by the two dominant poles using the transfer function

$$G(s) = \frac{1}{(s - p_1)(s - p_2)}$$
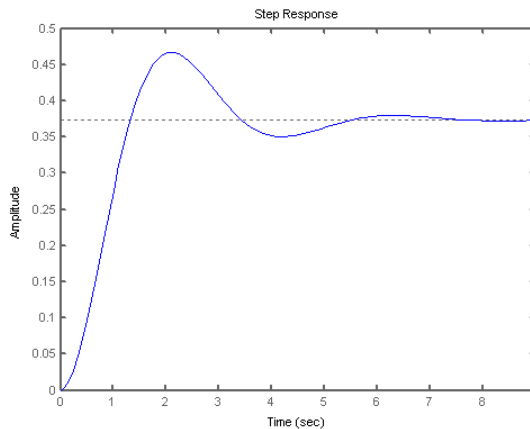
Step Response

HOWEVER: Using the step function on matlab we can see what the step response is for our system.

As shown above, the approximation is NOT valid for our system.
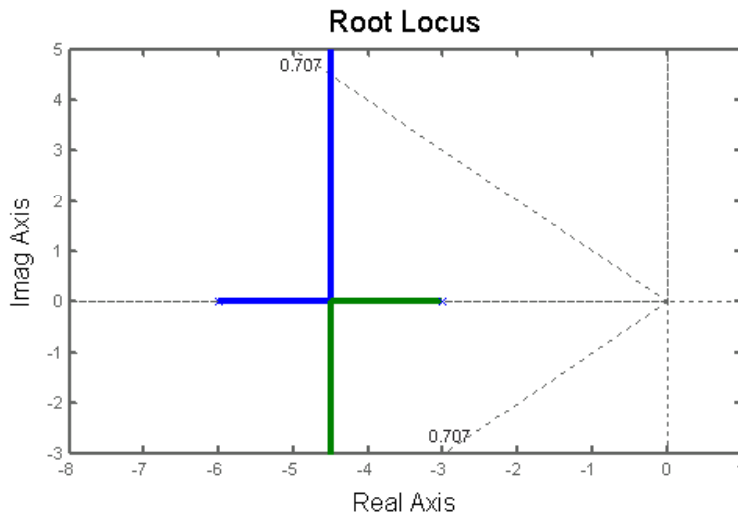
Problem 2   Nise 9 – 1

Design a PI controller to drive the step response to zero for the unity feedback system with transfer function:

$$G(s) = \frac{K}{(s+3)(s+6)}$$

The system should have a damping ratio:
$$\zeta = .707$$

Compare compensated and uncompensated systems. Quickly sketching the root locus, it should look like:

Now you should know that $\zeta = .707$ m has a slope of -1 as shown in the picture. Given this simple geometry you can find the closed loop poles at this point to be

$$p_{1,2} = -4.5 \pm 4.5i$$

To drive the steady state error to zero we need to add a pole at zero. In order to not change the transient response we add a zero close to the pole in order to cancel it out. I am placing the zero at -0.3, just so that you see the effect of the zero, if instead I had placed it for instance at -0.01 the effect would be much smaller.

We choose the PI controller to be:
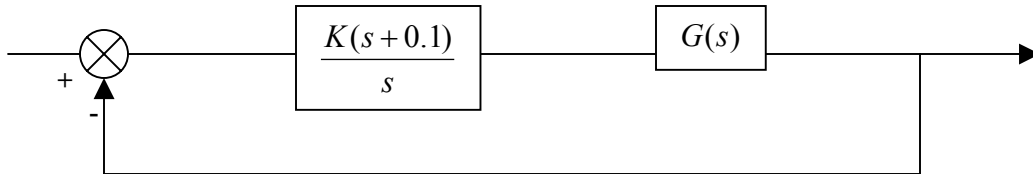
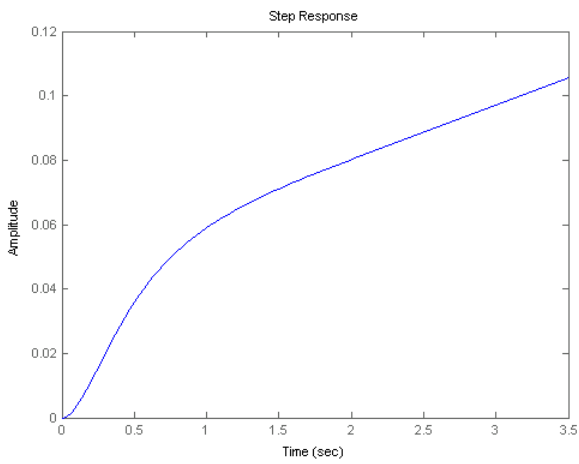Now the new system we are analyzing is:

$$\frac{K(s+0.1)}{s(s+3)(s+6)}$$

We plot this system into matlab and again find the point where the root locus intersects $\zeta = 0.707$ at the poles $p_{1,2} = -4.41 \pm 4.42i$

Now we can compare the response for both the compensated and uncompensated systems:

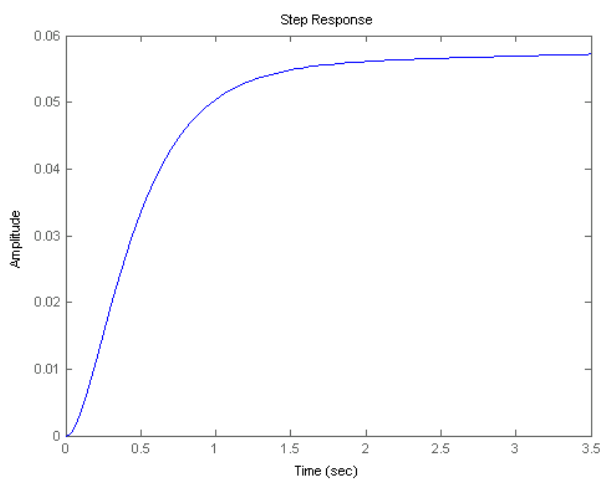| Characteristic	Uncompensated	Compensated
Poles	$p_{1,2} = -4.5 \pm 4.5i$	$p_{1,2} = -4.41 \pm 4.42i$
Gain at poles	$K = 22.5$	$K = 22.5$
ζ	0.707	0.707
T_s	$\frac{4}{4.5} = 0.89\,\text{sec}$	$\frac{4}{4.41} = 0.91\,\text{sec}$
K_p	$\frac{22.5}{18} = 1.25$	∞
$e(\infty)$	0.44	0
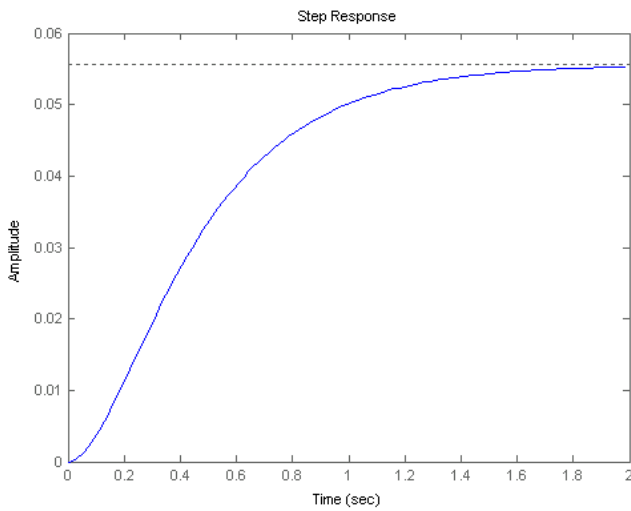
|	
	Step Response for the Uncompensated system is:
	Step Response for the Compensated system with zero at −0.3. Not very good pole-zero cancellation
	Step Response for the Compensated system with zero at −0.01 is: Good pole/zero cancellation

Problem 3   Nise 9-6

$$G(s) = \frac{K}{(s+1)(s+2)(s+3)(s+6)}$$

Operates dominant poles at $\zeta = 0.707$.

Design a PD controller so that the transient response will be reduced by half. Use Matlab to plot the root locus.
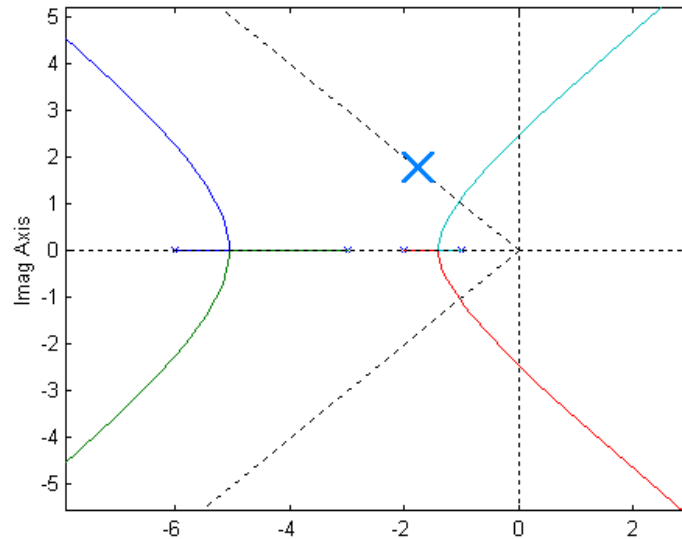
Using matlab we can find the point of intersection between the damping ratio and our closed loop poles. Namely at a gain of $K = 16.62$, $p_{1,2} = -1.04 \pm 1.04i$ while the higher order poles are at $p_3 = -4.29$ and $p_4 = -5.62$.

Now in order to reduce the settling time, $T_s = \dfrac{4}{\zeta \omega_n}$, by a factor of 2, we need to increase $\zeta \omega_n$ by a factor of two. So we know our new pole location is: $p_{1,2} = -2.08 \pm 2.08i$. The large x in the root locus shows this pole position. We are now ready to impose PD control. We have to place a zero along the real axis that will force the root locus to include the pole that we have found. In order to do this we know that the sum of the angles from our chosen point to the open loop poles and zeros are
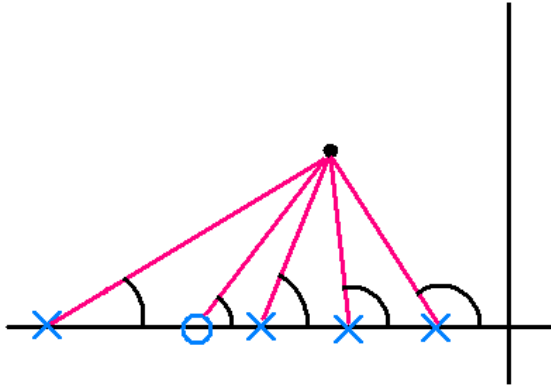
$$\theta_{p_1} + \theta_{p_2} + \theta_{p_3} + \theta_{p_4} - \theta_{z_c} = 180$$

$$\theta_n = \tan^{-1}\left(\frac{2.08}{-2.08 - p_n}\right),$$

where p_n is the pole i.e. $p_1 = -1$

$$117.4° + 92.2° + 66.1° + 28.0° - 180 = \theta_{z_c}$$

$$\theta_{z_c} = 123.7$$

Now that we know the angle for zero we know that our guess was incorrect, since the angle is greater than 90, but regardless we can still get the correct answer.
The zero location is given by:

$$\frac{2.08}{2.08 - z_c} = \tan(180 - \theta_{z_c})$$

$$z_c = 0.69$$
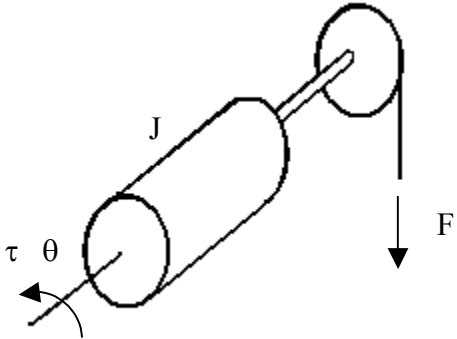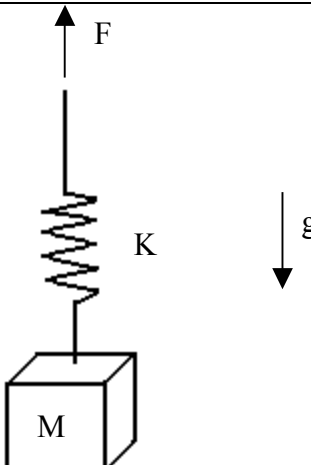
$$G_c = \frac{K(s + 0.69)}{(s + 1)(s + 2)(s + 3)(s + 6)}$$

Characteristic	Uncompensated	Compensated
Poles	$p_{1,2} = -1.04 \pm 1.04i$	$p_{1,2} = -2.08 \pm 2.08i$
Gain at poles	$K = 16.62$	$K = 19.78$
ζ	0.707	0.707
T_s	$\dfrac{4}{1.04} = 3.85\,\text{sec}$	$\dfrac{4}{2.08} = 1.93\,\text{sec}$
K_p	$\dfrac{16.62}{36} = 0.46$	$\dfrac{19.78 \cdot 0.69}{36} = 0.38$
$e(\infty)$	0.68	0.72

The problem is that the steady state error has increased.

Problem 4

a)   Solving the equations of motion

		These above are the equations of motion. Now we can take the Laplace transform to find the transfer functions

$J\ddot{\theta} = \tau - FR$ $J\ddot{\theta} = \tau - RK(R\theta - y)$	$M\ddot{y} = KF$ $M\ddot{y} = K(R\theta - y)$
$(Js^2 + R^2K)\theta = \tau + RKy \rightarrow \theta = \dfrac{\tau + RKy}{Js^2 + R^2K}$ Here we replace y on the left equation to get the transfer function between τ and θ. $(Js^2 + R^2K)\theta = \tau + RK\left(\dfrac{RK\theta}{Ms^2 + K}\right)$ $\dfrac{Js^2(ms^2 + K) + R^2K(ms^2 + K) - R^2K^2}{ms^2 + K} = \tau$ $\dfrac{\theta}{\tau} = \dfrac{ms^2 + K}{s^2\left(JMs^2 + (KJ + RKM)\right)}$ Two zeros, Four Poles	$Ms^2y + Ky = KR\theta \rightarrow y = \dfrac{KR\theta}{Ms^2 + K}$ Now we replace θ on the equation on the left to get the transfer function between τ and y $Ms^2y + Ky = RK\left(\dfrac{\tau + RKy}{Js^2 + R^2K}\right)$ $(Ms^2y + Ky)(Js^2 + R^2K) = RK(\tau + RKy)$ $s^2\left(MJs^2 + (KJ + MR^2K)\right)y(s) = RK\tau(s)$ $\dfrac{y(s)}{\tau(s)} = \dfrac{RK}{s^2\left(MJs^2 + (KJ + MR^2K)\right)}$ NO zeros, Four Poles

This is another example of collocated vs non-collocated systems. Just like the quiz question and the example done in class. There are two inertias connected by a spring. After PD control is applied you can see that when the control and sensor are on opposite sides (as in when you apply a torque and measure the position of the mass) you will get a very unstable system. Whereas when the control and sensor are on the same side, a stable

system will result. The root locus clearly illustrates this even though we don't have exact values available.

$$\frac{\theta}{\tau} = \frac{ms^2 + K}{s^2\left(JMs^2 + (KJ + RKM)\right)}$$

Two zeros, Four Poles

Applying PD control we have a zero at $-k_d$

$$\frac{y(s)}{\tau(s)} = \frac{RK}{s^2\left(MJs^2 + (KJ + MR^2K)\right)}$$

NO zeros, Four Poles