Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics and Control III
Spring 2002

SOLUTIONS: Problem Set # 9

Problem 1
For each of the following transfer functions:

a)
i) Please pay attention to the process for obtaining the magnitude and phase response from a transfer function. I will go in most detail for this first example.Given:

$$G(s) = \frac{10}{s(s+2)(s+5)}$$

First, by convention we express each factor of the transfer function as $(\tau s + 1)$, where the break frequency will be $\frac{1}{\tau}$. This will force the magnitude of each term to be *1* as the frequency approaches 0. This is convenient when plotting the bode asymptote approximation for each part of the transfer function. When they are expressed in this format, they will begin at 0 dB before the break point. As can be seen in the plot attached. If instead they were left in the form given by the transfer function. Then each contribution would have to be shifted by the gain at small frequencies.

$$G(s) = \frac{10}{10s\left(\frac{s}{2}+1\right)\left(\frac{s}{5}+1\right)}$$

Now we plug in $j\omega$ for s

$$G(j\omega) = \frac{1}{j\omega\left(\frac{j\omega}{2}+1\right)\left(\frac{j\omega}{5}+1\right)}$$

Now, the total magnitude of G can be found by multiplying out the denominator, grouping by *Im* and *Re* parts, then taking the complex conjugate of that so that we can express everything as a complex number. We DON'T WANT to do this. Several of you did it in the problem set, but that is not the point. Instead you want to take advantage of the fact that the total magnitude of $|G(j\omega)|$ is the product of the magnitude of the individual contributions:

$$|G(j\omega)| = \frac{1}{|j\omega|\left|\frac{j\omega}{2}+1\right|\left|\frac{j\omega}{5}+1\right|}$$

$$|G(j\omega)| = \frac{1}{\omega\left(\sqrt{\frac{\omega^2}{4}+1}\right)\left(\sqrt{\frac{\omega^2}{25}+1}\right)}$$

Now, by convention we express the magnitude in terms of $20\log|G(j\omega)|$:

$$\boxed{20\log|G(j\omega)| = -20\log\omega - 20\log\sqrt{\frac{\omega^2}{4}+1} - 20\log\sqrt{\frac{\omega^2}{25}+1}}$$

Notice that now all these responses can be simply added together. No need to divide/multiply.

Similarly we could also find the phase if we express $G(j\omega)$ as a complex number, but we take advantage of the fact that the phase will be given by the sum of the angular contribution of the zeros minus the sum of the angular contribution of the poles:

$$\angle G(j\omega) = -\angle(j\omega) - \angle\left(\frac{j\omega}{2}+1\right) - \angle\left(\frac{j\omega}{5}+1\right)$$

$$\boxed{\angle G(j\omega) = -90° - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{5}}$$

ii) Bode Plot:

Here you are asked to plot asymptotic sketches of the magnitude and frequency response. The results are attached. Notice that you should have straight lines for each of the approximations. The plots are made are in color, if they are not clear please take a look at them on the web.

iii) Break Points

Note that you were able to predict the shape of the Bode plot through the asymptotic sketches. These are the actual values at each of the breakpoints which would help you improve the accuracy of your sketch.

ω	$\|G(j\omega)\|$	$20\log\|G(j\omega)\|$
2	0.3283	-9.68 dB
5	0.0525	-25.59 dB

b)

$$G(s) = \frac{(s+10)}{(s+1)(s+50)}$$

$$G(s) = \frac{10\left(\frac{s}{10}+1\right)}{50(s+1)\left(\frac{s}{50}+1\right)}$$

$$G(j\omega) = \frac{\left(\frac{j\omega}{10}+1\right)}{5(j\omega+1)\left(\frac{j\omega}{50}+1\right)}$$

$$|G(j\omega)| = \frac{\left|\frac{j\omega}{10}+1\right|}{5|j\omega+1|\left|\frac{j\omega}{50}+1\right|}$$

$$\boxed{20\log|G(j\omega)| = 20\log\sqrt{\frac{\omega^2}{100}+1} - 20\log 5 - 20\log\sqrt{\omega^2+1} - 20\log\sqrt{\frac{\omega^2}{50^2}+1}}$$

Notice that the constant gain of 1/5 contributes to the magnitude response, but NOT the phase respose.

$$\angle G(j\omega) = \angle\left(\frac{j\omega}{10}+1\right) - \angle(j\omega+1) - \angle\left(\frac{j\omega}{50}+1\right)$$

$$\boxed{\angle G(j\omega) = \tan^{-1}\frac{\omega}{10} - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{50}}$$

ii) Bode Plot:

Attached

iii) Break Points:

ω	$\lvert G(j\omega)\rvert$	$20\log\lvert G(j\omega)\rvert$
1	0.1421	-16.95 dB
10	0.0276	-31.18 dB
50	0.0144	-36.82 dB

c)

$$G(s) = \frac{s(s+30)}{(s^2+5s+25)}$$

Note that even though this is a second order system, we also express it so that the gain is 1 when the frequency approaches 0.

$$G(s)=\frac{30s\left(\frac{s}{30}+1\right)}{25\left(\frac{s^2}{25}+\frac{s}{5}+1\right)}$$

$$G(j\omega)=\frac{30s\left(\frac{j\omega}{30}+1\right)}{25\left(-\frac{\omega^2}{25}+\frac{j\omega}{5}+1\right)}$$

$$\boxed{20\log\left|G(j\omega)\right|=20\log\frac{6}{5}+20\log\omega+20\log\sqrt{\frac{\omega^2}{30^2}+1}-20\log\sqrt{\left(1-\frac{\omega^2}{25}\right)^2+\frac{\omega^2}{25}}}$$

$$\boxed{\angle G(j\omega)=\angle(\omega)+\angle\left(\frac{j\omega}{30}+1\right)-\angle\left(\left(1-\frac{\omega^2}{25}\right)+j\frac{\omega}{5}\right)}$$

ii) Bode Plot:

Attached

iii) Break Points:

ω	$\left|G(j\omega)\right|$	$20\log\left|G(j\omega)\right|$
5	6.0828	16.01 dB
30	1.4337	3.13 dB

Plotting bode plots Step by Step:

First:
Express each pole and zero as $[\tau s+1]$, for the single poles/zeros and $\left[\left(\frac{s}{\omega}\right)^2+2\zeta\left(\frac{s}{\omega}\right)+1\right]$ for second order (complex) poles/zeros. When doing so a constant gain may be factored out, (as in problem b and c). This will be the DC gain of the bode plot.

Second:
Find the break frequency for each pole and zero. Single poles/zeros will break at $1/\tau$. Complex pole/zeros will break at ω. Pure poles and zeros: $1/s$, s, will not break (see problem a and c).

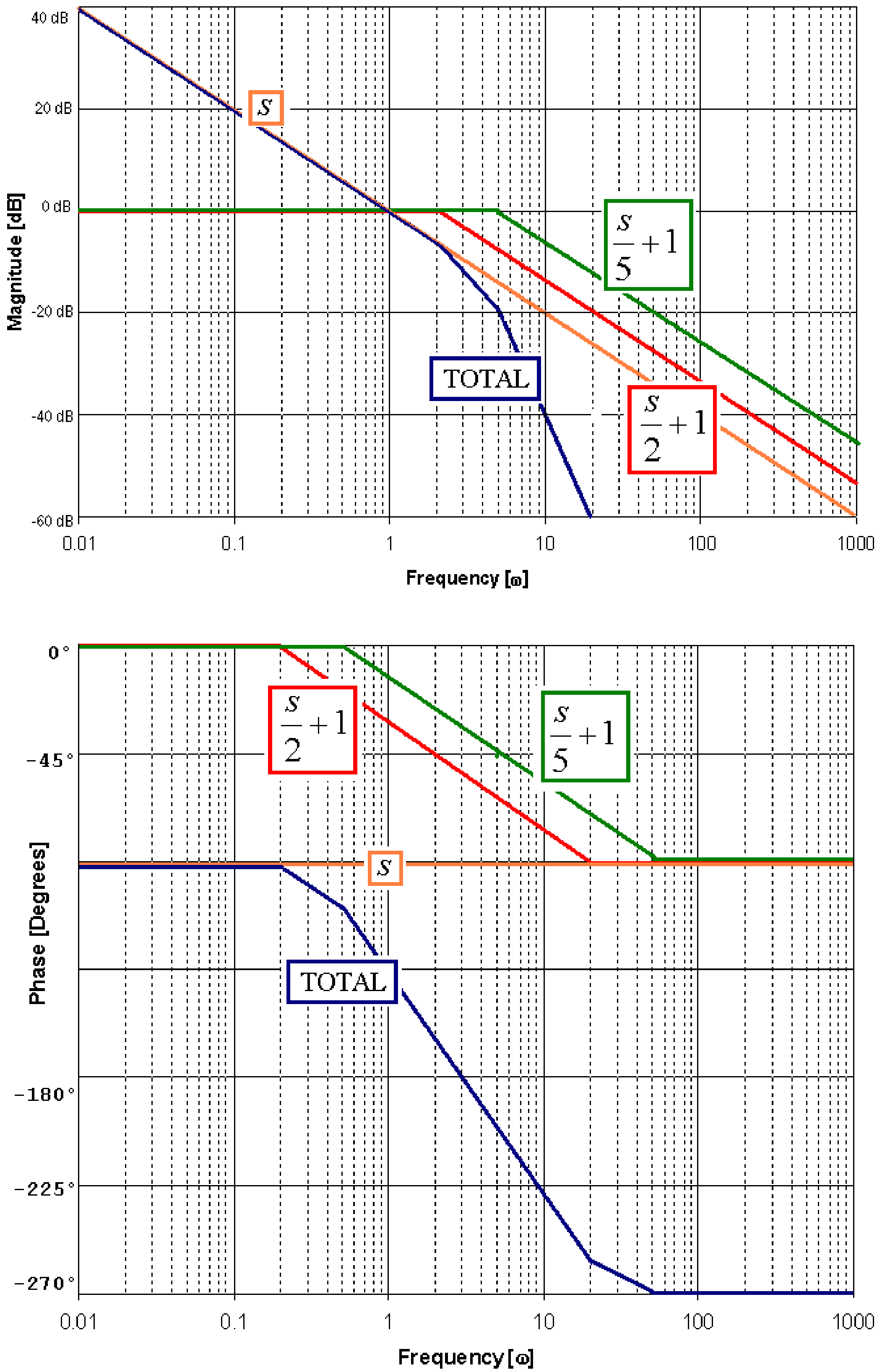
Third:
Magnitude
Plot the asymptotic magnitude response for each pole/zero.
For a **simple pole/zero** the plot will start at 0 dB and continue until the break frequency where poles will have a slope of –20 dB/dec and zeros a slope of +20 dB/dec.
For a **complex pole/zero**, the plot will again start at 0 dB and continue until the break frequency where complex poles will have a slope of –40 dB/dec and complex zeros a slope of +40 dB/dec.
Pure poles will have slope of –20 dB/dec and go through the 0 dB at a frequency of 1
Pure zeros will have slope of +20 dB/dec and go through the 0 dB at a frequency of 1.
The **DC gain** will shift the total response up or down, converted by 20 log (gain).
Phase
Plot the asymptotic phase response for each pole/zero.
For a **simple pole** at the break frequency the phase will be –45°, one decade earlier the phase will be 0°, one decade later it will be –90°.
For **a simple zero** at the break frequency the phase will be +45°, one decade earlier the phase will be 0°, one decade later it will be +90°.
For a **complex pole** at the break frequency the phase will be –90°, one decade earlier the phase will be 0°, one decade later it will be –180°.
For a **complex zero** at the break frequency the phase will be +90°, one decade earlier the phase will be 0°, one decade later it will be +180°.
Pure pole will have a phase of –90°, a **pure zero** will have a phase of +90° for all ω.
DC gain – NO contribution

Fourth:
Sum up the contribution from each pole and zero to get the slope of the total response for both the magnitude and phase.

NOTE: This gives you a pretty good approximation of what the bode plot will look like. HOWEVER: For a second order pole/zero with ζ very small. It will cause the magnitude response to shoot up/down. So the asymptote will not be a good approximation.

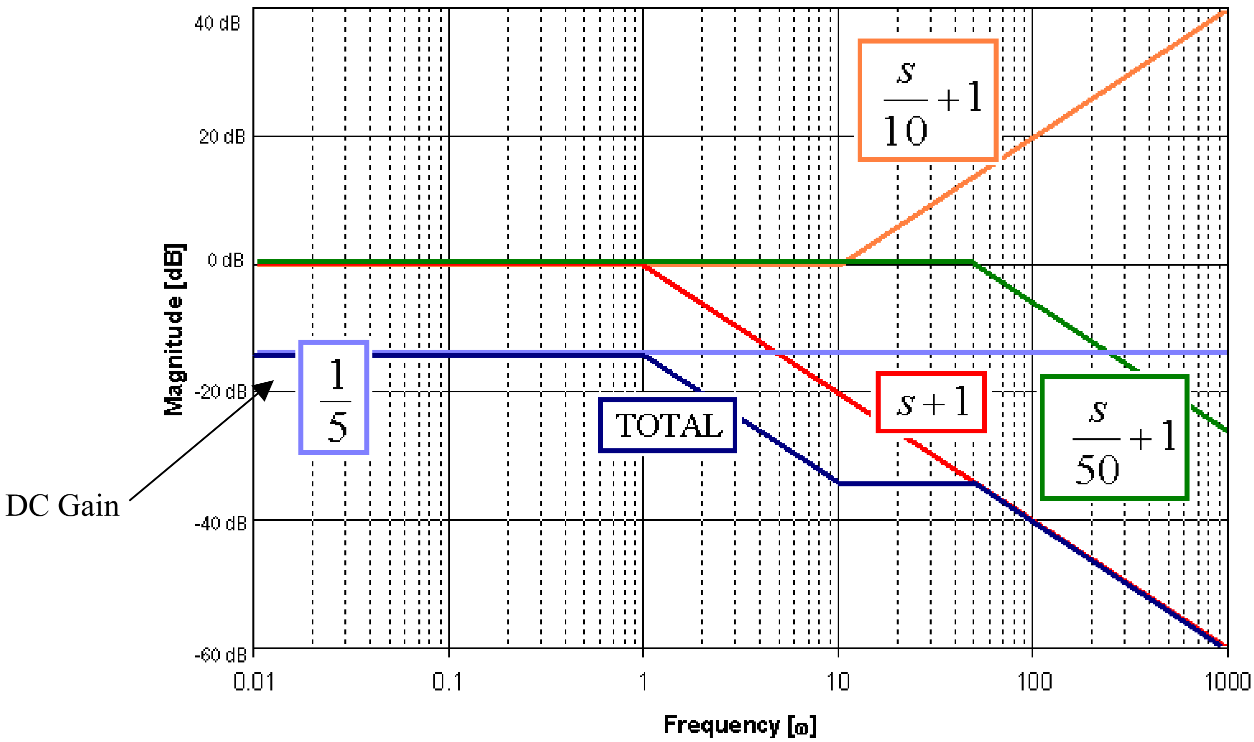
a)

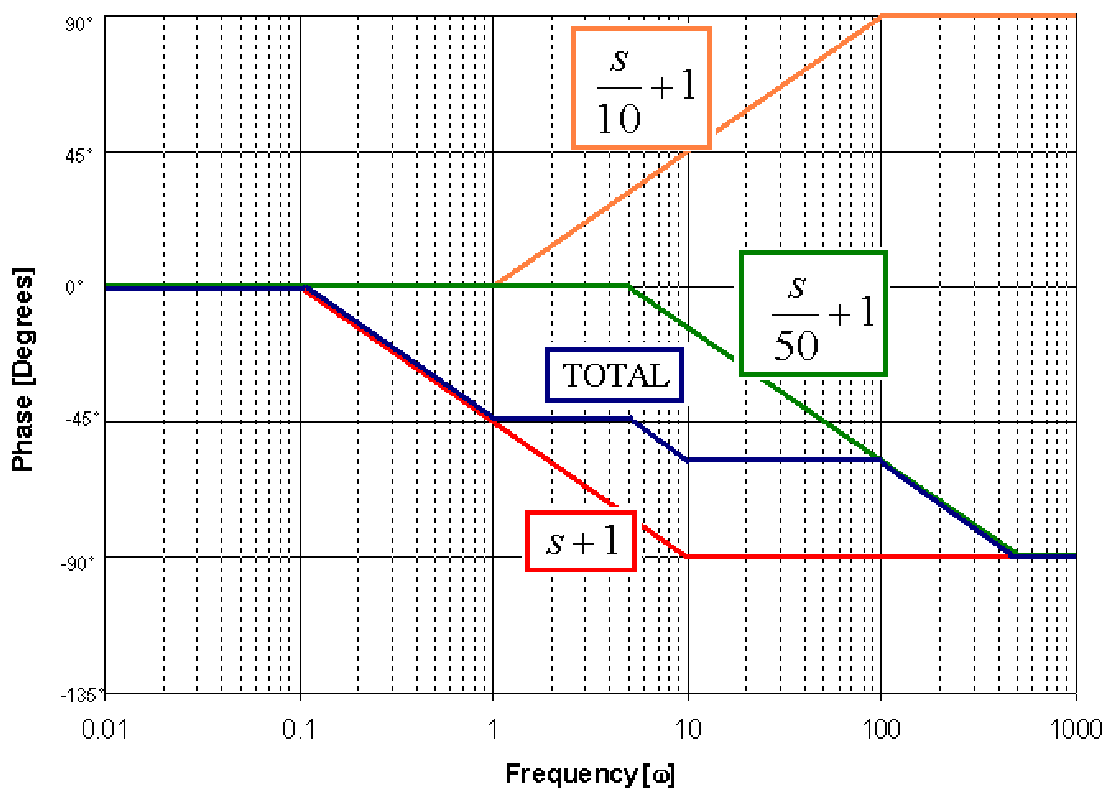
$$G(s) = \frac{10}{s(s+2)(s+5)}$$



b)

$$G(s) = \frac{(s+10)}{(s+1)(s+50)}$$





c)

$$G(s) = \frac{s(s+30)}{(s^2+5s+25)}$$