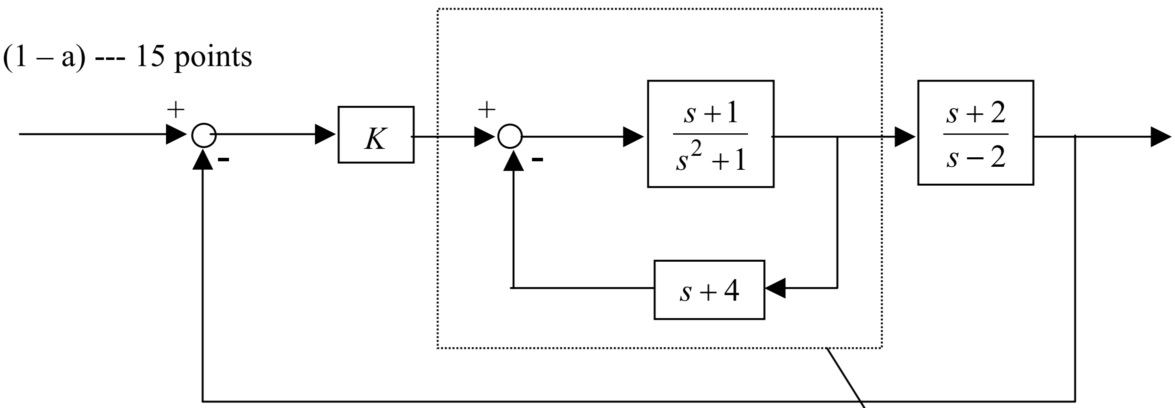
Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics, and Control III
Spring 2002

SOLUTIONS: Quiz #1

Problem 1

(1 – a) --- 15 points



Reducing the block diagram:

$$G(s) = \frac{\frac{s+1}{s^2+1}}{1+\frac{s+1}{s^2+1}s+4}\,\frac{s+2}{s-2}$$

$$\frac{\frac{s+1}{s^2+1}}{1+\frac{s+1}{s^2+1}(s+4)}$$

$$G(s) = \frac{s+1}{s^2+1+(s+1)(s+4)}\,\frac{s+2}{s-2}$$

$$\boxed{G(s) = \frac{(s+1)(s+2)}{(2s^2+5s+5)(s-2)}}$$

(1 – b) ----- 15 points

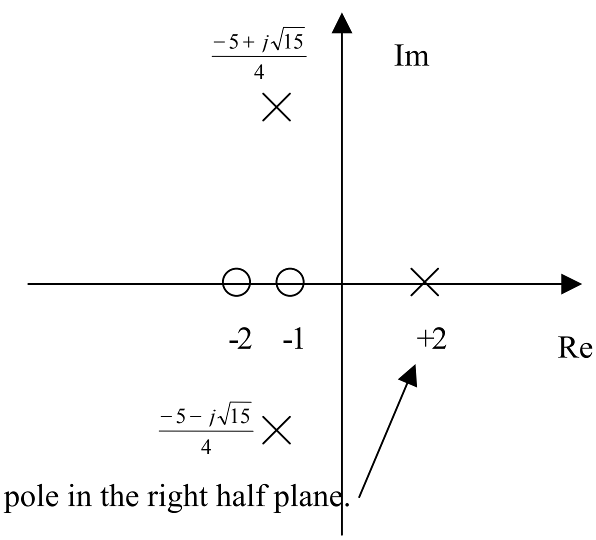
Open Loop Poles and zeros

Poles: $2s^2+5s+5=0$

$$p_{1,2} = \frac{-5\pm\sqrt{25-40}}{4} = \frac{-5\pm j\sqrt{5}}{4} \cong -1.25\pm j1$$

$p_3 = +2$

Zeros: $z_1=-1,\ z_2=-2$



The open loop system is unstable, since there is one pole in the right half plane.

(1 – c) ----- 20 points

The closed-loop transfer function is given by:

$$T(s) = \frac{KG}{1+KG}$$

$$T(s) = \frac{K(s+1)(s+2)}{(2s^2+5s+5)(s-2)+K(s+1)(s+2)}$$

$$T(s) = \frac{K(s+1)(s+2)}{2s^3+(K+1)s^2+(3K-5)s+(2K-10)}$$

Characteristic equation:

$$2s^3+(K+1)s^2+(3K-5)s+(2K-10)=0$$

The Routh Hurwitz Table:

s^3	2	3K-5
s^2	K+1	2K-10
s^1	b_1	0
s^0	2K-10	0

Where,

$$b_1 = \frac{(3K-5)(K+1)-2(2K-10)}{K+1}$$

$$= \frac{3(K^2-2K+5)}{K+1}$$

For stability all must be greater than zero

- $K+1>0 \rightarrow K>-1$
- $2K-10>0 \rightarrow K>5$
- $K^2-2K+5>0 \rightarrow (K-1)^2+4>0$

Therefore this condition is met for all real *K*.

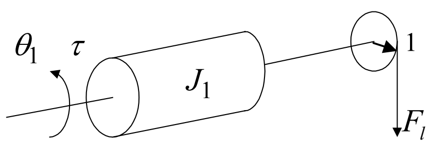
$\therefore$ Stability Condition: K > 5

Problem 2

(2 – a) ----- 15 points

Free-body diagrams

$$J_1\ddot{\theta}_1 = \tau - F_l \times 1$$



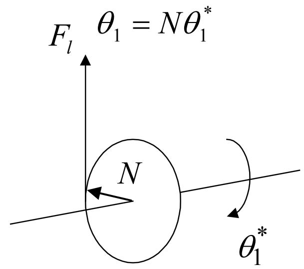
$$J_1 s^2 N\theta_1^* = \tau - \frac{K_t}{N}(\theta_1^* - \theta_2)$$

$$J_1 s^2 N^2\theta_1^* = N\tau - K_t\theta_1^* + K_t\theta_2$$

$$(J_1 s^2 N^2 + K)\theta_1^* = N\tau - K_t\theta_2$$

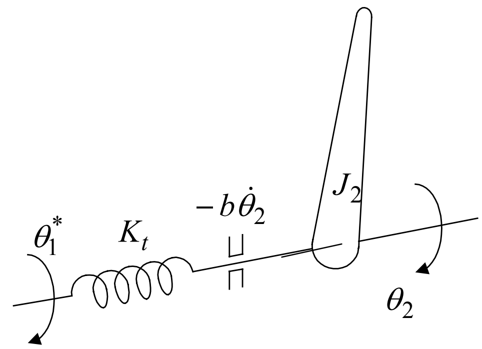
$$\theta_1^* = \frac{N\tau + K_t\theta_2}{J_1 s^2 N^2 + K_t}$$

$$0 = F_l \cdot N - K_t(\theta_1^* - \theta_2)$$



$$F_\ell = \frac{K_t}{N}(\theta_1^* - \theta_2)$$

$$J_2\ddot{\theta}_2 = -b\dot{\theta}_2 + K_t(\theta_1^* - \theta_2)$$



$$(J_2 s^2 + bs + K)\theta_2 = K_t\theta_1^*$$

$$(J_2 s^2 + bs + K_t)\theta_2 = \frac{K_t(N\tau + K_t\theta_2)}{J_1 s^2 N^2 + K_t}$$

and output y is given by

$$y = L\theta_2$$

Solving for y yields:

$$G(s) = \frac{y(s)}{\tau(s)} = \frac{K_t L N}{s[N^2 J_1 J_2 s^3 + N^2 J_1 b s^2 + K_t(N^2 J_1 + J_2)s + K_t b]}$$

(2 – b) ----- 20 points

Obtaining the closed-loop transfer function,

$$T(s) = \frac{k_p G}{1 + k_p G} = \frac{5k_p}{s(s^3 + 12s^2 + 66s + 132) + 5k_p}$$

The characteristic equation is:

$$C.E. = s^4 + 12s^3 + 66s^2 + 132s + 5k_p = 0$$

Using the Routh table:

s^4	1	66	$5k_p$
s^3	~~12~~ → 1	~~132~~ → 11	0
s^2	55 → 11	$5k_p \rightarrow k_p$	0
s^1	$\frac{121-k_p}{11}$	0	0
s^0	k_p	0	0

For positive k_p the elements of the first column in the Routh table are all positive except for s^1. The closed-loop system becomes marginally stable when this coefficient becomes zero:

$$\frac{121-k_p}{11} = 0, \quad \therefore k_p = 121$$

When $k_p = 121$ the row s^1 becomes all zero; this will make the coefficients of s^2 be a factor of the characteristic equation.

$$11s^2 + k_p = 0$$

$$s = \pm j\sqrt{\frac{k_p}{11}} = \pm j\sqrt{11}$$

The oscillatory frequency is:

$$\omega_n = \sqrt{11} \ rad/s$$

(2 – c) ----- 15 points

The poles closest to the imaginary axis $p_{1,2}$ dominate the response

$p_{1,2} = -1 \pm j2$

$$G(s) = \frac{y(s)}{\tau(s)} = \frac{K_t LN}{s[N^2 J_1 J_2 s^3 + N^2 J_1 b s^2 + K_t(N^2 J_1 + J_2)s + K_t b]}$$

$\omega_d = 2$, $\sigma = 1$

$$(s+1-j2)(s+1+j2) = s^2 + 2s + 5$$

Settling time:

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} = 4\sec$$