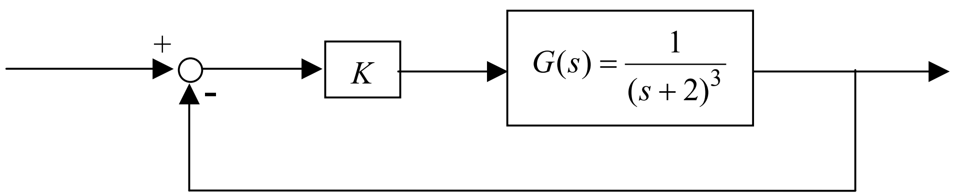
Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics, and Control III
Spring 2002

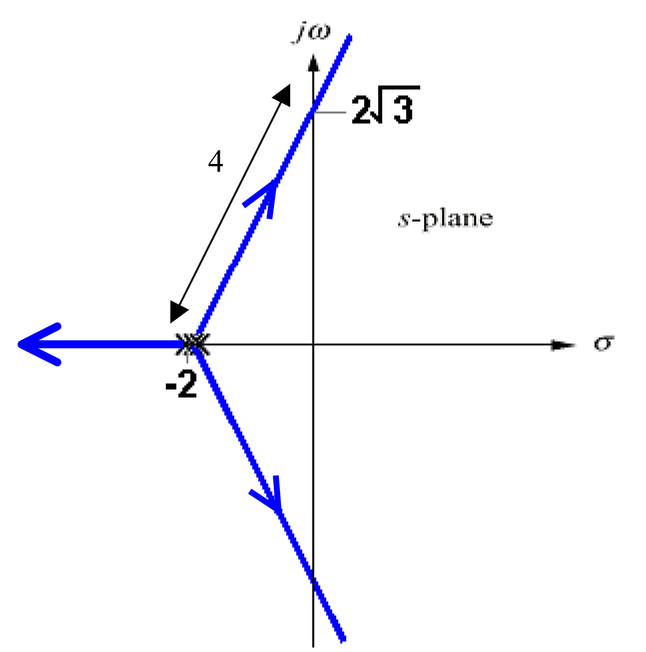
SOLUTIONS: Quiz #2

Problem 1

(1 – a) [10 points] Plotting the Root Locus



Root Locus



$$\angle \frac{1}{(s+2)^3} = 180°$$

$$3\angle(s+2) = -180° + 360° K$$

$$\angle(s+2) = 60°, -60°, 180°$$

Three repeated poles are at –2. Note that all the root locus branches go straight from –2 to the three directions. So the three asymptotes give the exact root locus.

(1 – b) [15 points] Feedback Gain

$$K = \frac{1}{|G(s)|} = |s+2|^3$$

From our plot we know that the marginally stable poles lie at: $\pm j2\sqrt{3}$

$$|s+2| = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4$$
$$\therefore K = 4^3 = 64$$

This computation is quite simple, if you realize that you can use the characteristics of a 30-60-90 triangle to find each length graphically.

Checking with the Routh Hurwitz Table

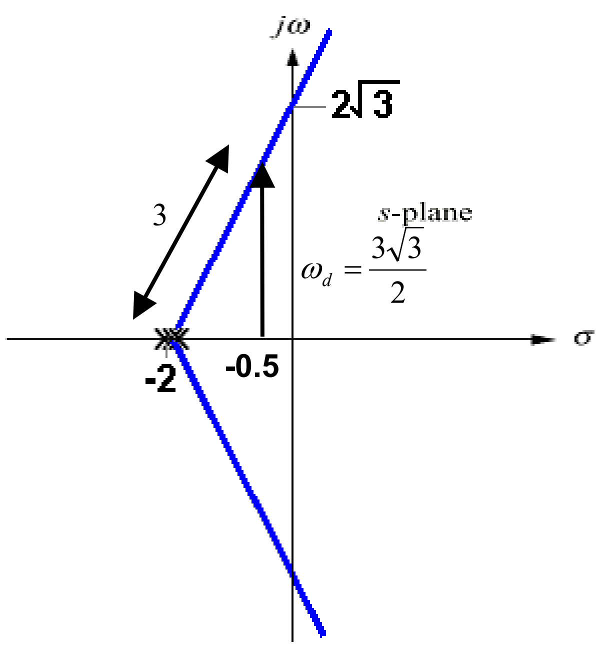
First finding the closed-loop characteristic equation:

$$T(s) = \frac{KG}{1+KG} = \frac{K}{(s+2)^3 + K} = \frac{K}{s^3 + 6s^2 + 12s + (8+K)}$$

s^3	1	12
s^2	6	8 + K
s^1	$12 - \frac{8+K}{6}$	0
s^0	8 + K	0

So, $8 + K < 72 \;\therefore K < 64$
Marginally stable when $K = 64$, just as we found graphically.

(1 – c) [15 points] Poles for peak time



Peak time given by:

$$T_p = \frac{2\sqrt{3}\pi}{9} = \frac{\pi}{\omega_d}$$

$$\omega_d = \frac{9}{2\sqrt{3}} = \frac{3\sqrt{3}}{2}$$

The position of this pole can be found graphically, using the characteristics of a 30-60-90 triangle as shown above. Or, you can also think of the root locus is given by the line with slope = $\sqrt{3}$, and y-intercept $2\sqrt{3}$. Then, the equation for this line would be:

$$y = \sqrt{3}x + 2\sqrt{3}$$

We already know the y-coordinate so all we need to find is the x-coordinate.

$$y = \frac{3\sqrt{3}}{2}, \text{ so:}$$

$$\frac{3\sqrt{3}}{2} = \sqrt{3}x + 2\sqrt{3}$$

$$\therefore x = -\frac{1}{2}$$

So, the dominant closed loop poles are

$$p_{1,2} = -\frac{1}{2} \pm j\frac{3\sqrt{3}}{2}$$

$$K = |s+2|^3 = \left(\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}\right)^3 = 3^3 = 27$$

The third pole for this given gain is found by working backwards from the equation above. Replacing *s* by a real number σ in the above equation,

$$K = |\sigma - (-2)|^3 = |\sigma + 2|^3 = 27$$

$$|\sigma + 2| = 3$$

$$\therefore \sigma = -5$$

The third pole is at –5; 10 times farther from the imaginary axis than the dominant poles. Therefore the second-order approximation is valid.

Problem 2

(2 – a) [10points] Modeling

The equation of motion:

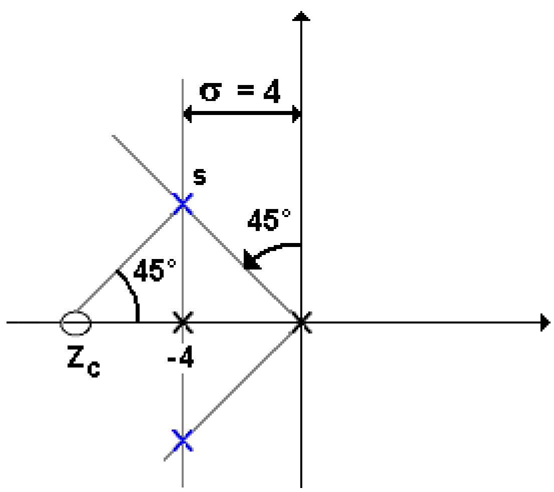
The nozzle reaction force creates this moment about the joint axis.

$$J\ddot{\theta} = \tau - b\dot{\theta} + F_d \times L$$

$$(Js^2 + bs)\theta(s) = \tau(s) + LF_d(s)$$

$$y(s) = \theta(s) \cdot L = \frac{L\tau(s) + L^2 F_d(s)}{s(Js + b)}$$

(2 – b) [20 points] PD Control



$$T_s = 1\sec = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

$$\sigma = 4$$

$$\zeta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$$

$$\theta = 45°$$

Now we know exactly what our desired Closed Loop Poles are:

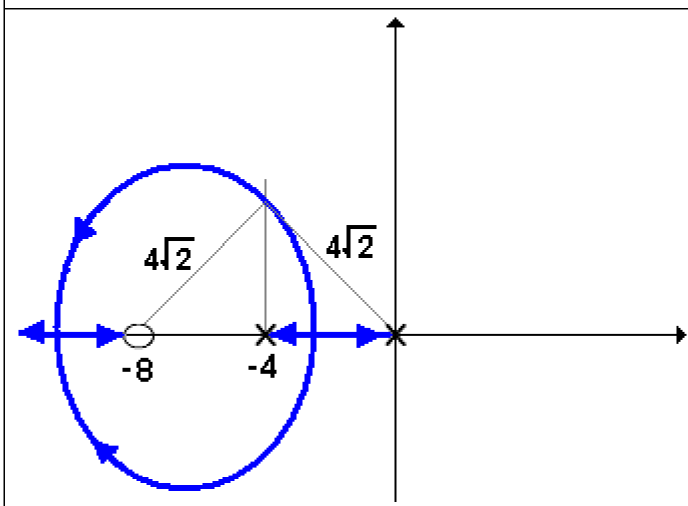
$$p_{1,2} = -4 \pm j4$$

$$\angle \frac{s + Z_c}{s(s+4)} = 180°$$

$$\angle(s + Z_c) = 180° + \angle s + \angle(s+4)$$

$$= 180° + 135° + 90° = 405°$$

$$= 45°$$



Now we need to find the location of the zero along the real axis

$$\frac{4}{Z_c - 4} = \tan 45° = 1$$

$$\therefore Z_c = 8$$

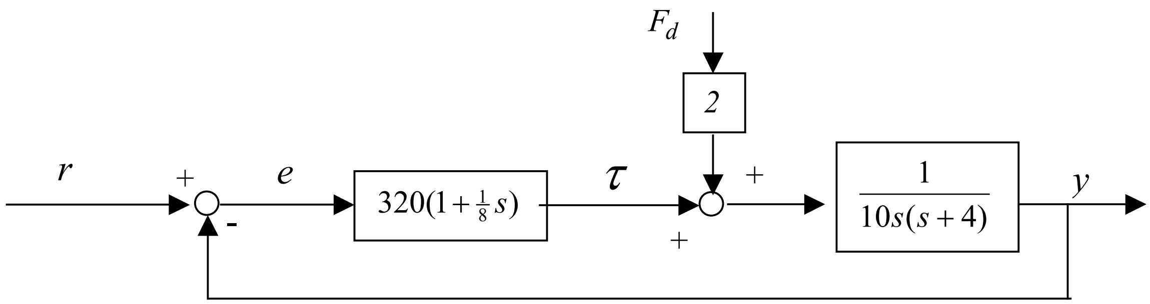
$$K'G_c G(s) = \frac{K'(s+8)}{10s(s+4)} = -1$$

$$K' = \frac{10|s| \cdot |s+4|}{|s+8|} = \frac{10 \cdot 4\sqrt{2} \cdot 4}{4\sqrt{2}} = 40$$

$$K'G_c = 40(s+8) = 320(1 + \tfrac{1}{8}s)$$

$$\therefore K = 320,\ k_D = \tfrac{1}{8} = 0.125$$

(2 – c) [15 points] Steady State error



Deriving the equation for error again:

$$y = G(2F_d + G_c e)$$
$$e = r - y = r - 2GF_d - GG_c e$$
$$(1 + GG_c)e = r - 2GF_d$$
$$e = \frac{r - 2GF_d}{1 + GG_c}$$

For $r = \frac{1}{s}$, $\quad e_{ss,r} = \lim_{s \to 0} s \frac{1}{s} \frac{10s(s+4)}{10s(s+4) + 320(1+\frac{1}{8}s)} = 0$

For $F_d = \frac{1}{s}$, $\quad -e_{ss,F_d} = \lim_{s \to 0} s \frac{1}{s} \frac{2}{10s(s+4) + 320(1+\frac{1}{8}s)} = \frac{1}{160} = 0.00625$

(2 – d) [15 points] Compensator: PI

In order to reduce the steady state error contribution of the disturbance to zero consider a PI controller with a pole at the origin and a zero at $s = -\sigma$

$$G_c = K\left(1 + \frac{1}{8}s\right)\left(1 + \frac{\sigma}{s}\right)$$

For $F_d = \frac{1}{s^2}$

$$-e_{ss,F_d} = \lim_{s \to 0} s \frac{1}{s^2} \frac{2}{10s(s+4) + 320\left(1+\frac{1}{8}s\right)\left(1+\frac{\sigma}{s}\right)}$$

$$-e_{ss}=\lim_{s\to 0}\frac{2}{10s^2(s+4)+320\left(1+\frac{1}{8}s\right)(s+\sigma)}=\frac{1}{160\sigma}=0.0625=6.25\%$$

$$\sigma=\frac{1}{160\times 0.0625}=0.1$$

The transfer function for the combined controller is now given by:

$$G_c=320\left(1+\frac{1}{8}s\right)\left(1+\frac{0.1}{s}\right)=\frac{(320+40s)(s+0.1)}{s}$$

$$G_c=\frac{10s^2+324s+32}{s}=32\frac{1.25s^2+10.125s+1}{s}$$