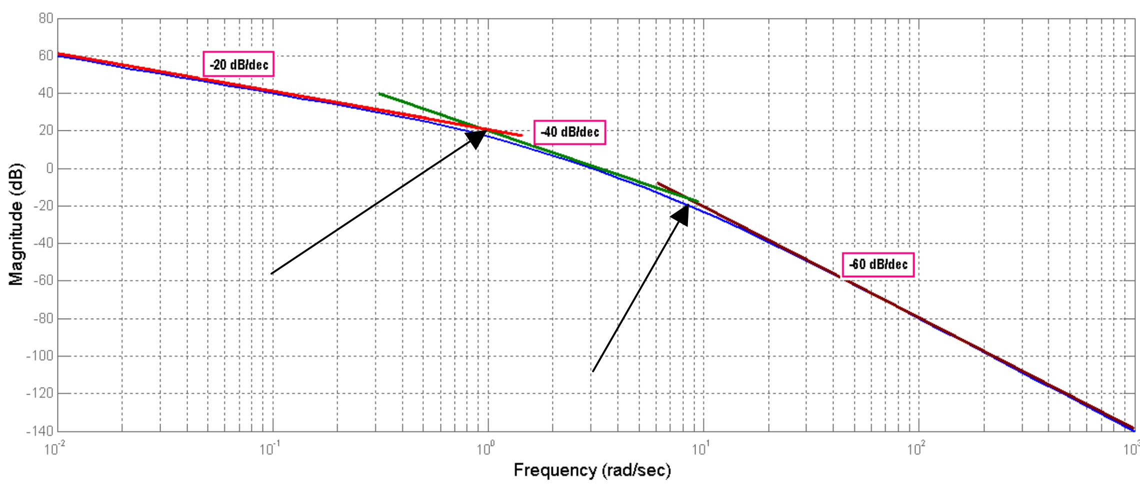
Department of Mechanical Engineering
Massachusetts Institute of Technology
2.010 Modeling, Dynamics, and Control III
Spring 2002

SOLUTIONS: Quiz #3

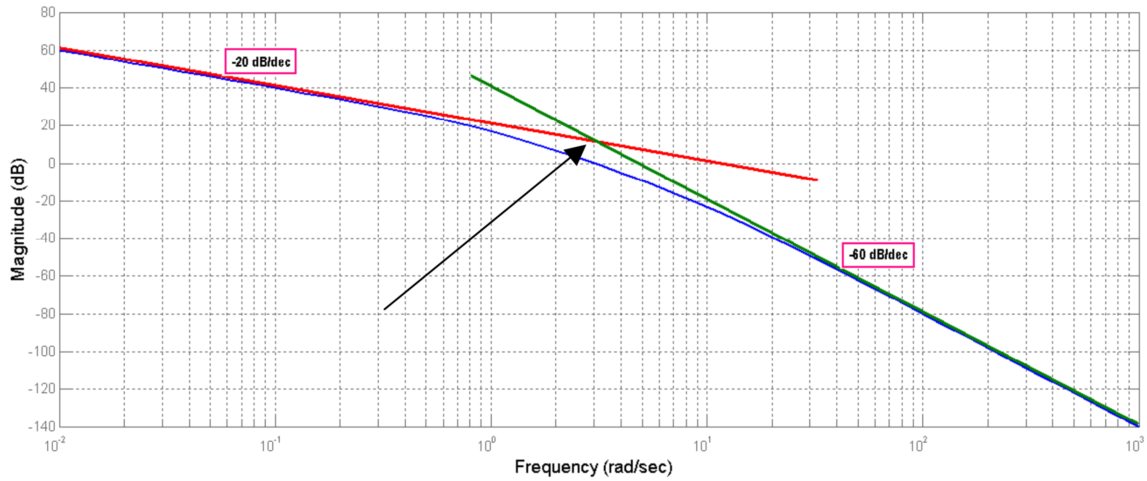
Problem 1
(1 – a) 15 points
The slope of the gain curve at low frequencies is –20 dB/dec; therefore there is one pole at the origin and this is a Type 1 system.
The phase curve asymptotically reaches –270°, hence the relative order of the system is 3. Since both the gain and the phase curves are monotonically decreasing, there is no zero in this system.

As mentioned, there is one pole at the origin, and there must be two more poles in the system.

Approximating the gain curve by three asymptotes of slope -20 dB/dec, -40 dB/dec and –60 dB/dec, respectively we obtain the break frequencies at 1 rad/sec and 10 rad/sec, as shown in the figure.



Another way of approximating the gain curve is to use two asymptotes of slope –20 dB/dec and –60 dB /dec, respectively. That is the case having one integrator and two repeated poles on the real axis. Then the intersection of these two asymptotes is found at 3 rad/sec. The break frequency is 3 rad/sec.



(1 – b) 10 points
At 3 rad/sec, the gain is 0 dB and the phase is –180°. Therefore the gain margin is 0 dB, and the phase margin is 0°.

$$GM = 0\,dB,\ \Phi_M = 0°$$

At this point the system is critically stable, i.e. the closed-loop poles are on the imaginary axis. Therefore, the undamped natural frequency is 3 rad/sec. Also note that $\Phi_M = \omega_n$, when $\varsigma = 0$.

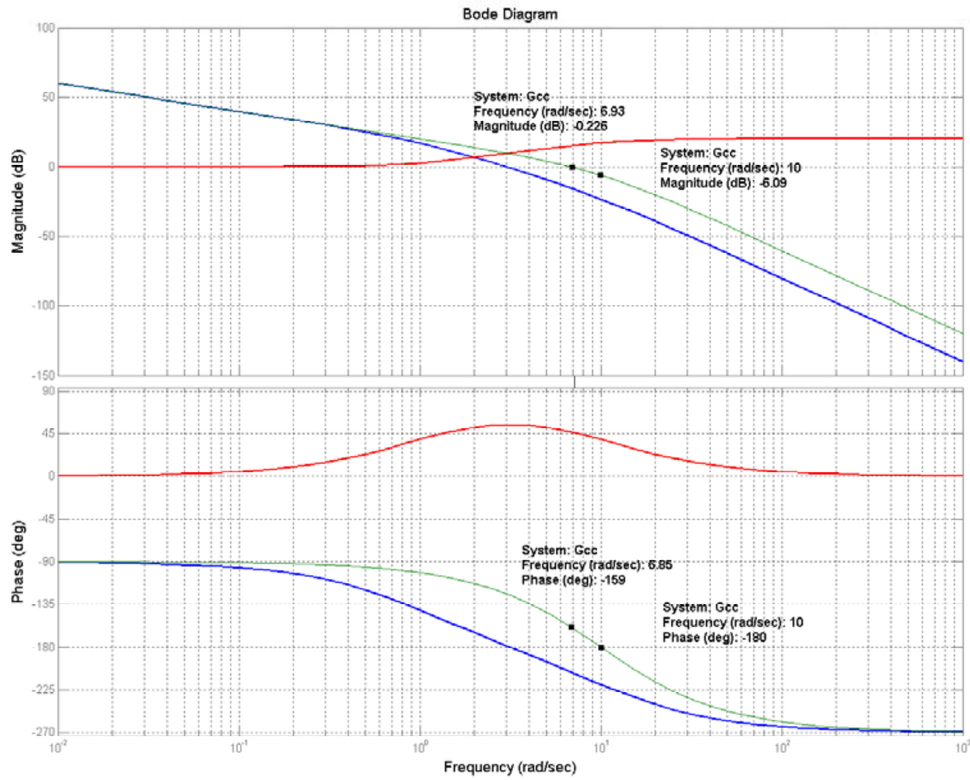
(1 – c) 10 points
See the attached Bode plots

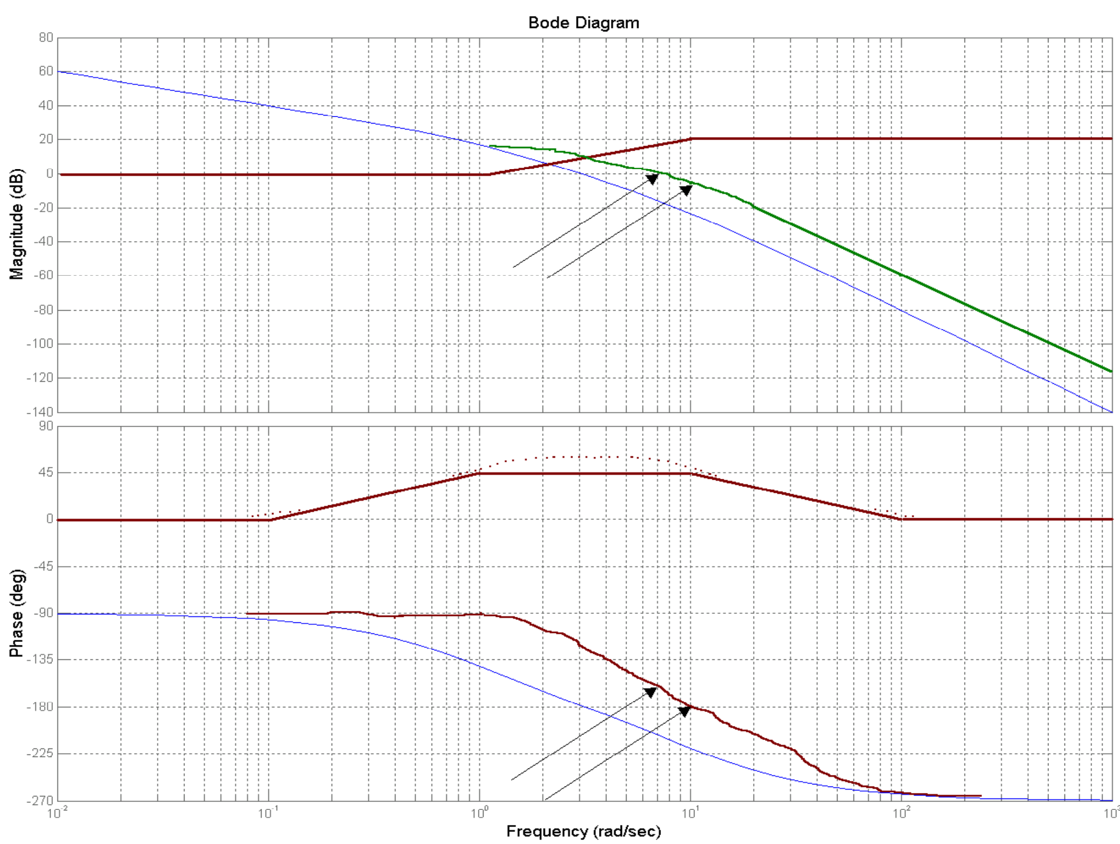
$$G_c = \frac{Ts+1}{\beta Ts+1} = \frac{s+1}{0.1s+1} \therefore T = 1,\ \ \beta = 0.1$$

(1 – d) 10 points
From the new compensated gain curve, the gain cross over frequency, i.e. the phase margin frequency, is 7 rad/s. From the phase curve the phase margin is 21°.

The compensated phase curve crosses the minus 180° line at 10 rad/s hence the gain margin is: 6 dB

Here is a plot of the exact solution done in matlab. For this problem, exact values are not needed, but your answer must be consistent with your plots.


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
System: Gcc
Frequency (rad/sec): 6.93
Magnitude (dB): -0.226
System: Gcc
Frequency (rad/sec): 10
Magnitude (dB): -6.09
System: Gcc
Frequency (rad/sec): 6.85
Phase (deg): -159
System: Gcc
Frequency (rad/sec): 10
Phase (deg): -180


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)

Obviously, MATLAB was not available on the test, so here is an approximated solution done by hand. The arrows show the same relevant points marked above.

(1 – e) 15 points

The damping ration ζ of the dominant poles is determined directly from the phase margin Φ_M. Approximately the relationship between ζ and Φ_M for $0 < \zeta \le 0.6$ is given by

$$\zeta \cong \frac{\Phi_M}{100} = 0.21$$

The percent overshoot is then

$$\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 50\%$$

The gain cross over frequency ω_{Φ_M} is related to ζ, and ω_n such that:

$$\omega_{\Phi_M} = \omega_n\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}$$

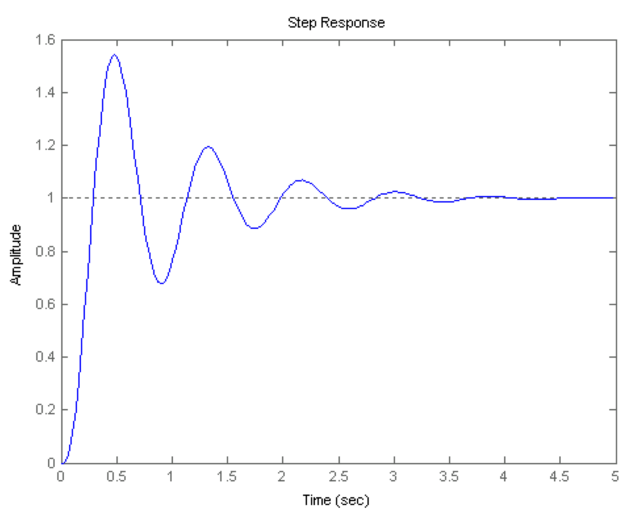
Therefore

$$\omega_n = \frac{\omega_{\Phi_M}}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} = 7.1\,\text{rad/s}$$

The peak time is obtained from ω_n and ζ

$$T_P = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.45\,\text{sec}$$

You didn't have to plot the step response, but it's nice to see that it does indeed it displays the characteristics that were calculated.



(1 – f) 10 points

The gain curve at low frequencies is approximated to:

$$|G(j\omega)| \cong \left|\frac{k}{s}\right|_{s=j\omega} = \frac{k}{\omega}$$

Velocity constant K_v is given by the frequency at which the extension of the approximate gain curve intersects the 0 dB line. Lifting the gain curve 10 dB higher, the approximate gain curve intersects the 0 dB line at 30 rad/sec, therefore $K_v = 30\,\text{rad/sec}$.

Problem 2
(2 – a) 15 points

This is a typical second order system, whose transfer function will look like:

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

Given a percent overshoot we can find the value of ζ :

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.46$$

The peak magnitude M_p is determined by ζ,

$$M_P = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.23$$

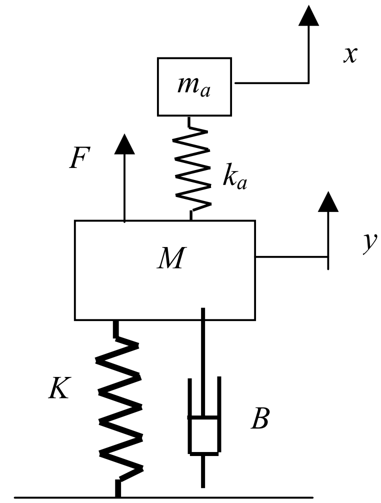
To obtain the peak resonant frequency the undamped natural frequency ω_n must be determined:

$$\omega_n = \sqrt{\frac{K}{M}} = 10 rad/\sec$$

Therefore the peak resonant frequency ω_r is

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} = 7.64 rad/\sec$$

(2 – b) 15 points



$$M\ddot{y} = -Ky - B\dot{y} + F + k_a(x - y)$$
$$m_a\ddot{x} = -k_a(x - y)$$
$$[Ms^2 + Bs + (K + k_a)]Y(s) = F + k_a x$$
$$[m_a s^2 + k_a]X(s) = k_a Y(s)$$

$$G(s) = \frac{Y(s)}{F(s)} = \frac{m_a s^2 + k_a}{[Ms^2 + Bs + (K + k_a)][m_a s^2 + k_a] - k_a^2}$$

The minimum for this transfer function well occur when $|G(j\omega)|$ becomes zero i.e. when the numerator of the transfer function becomes zero:

$$(m_a s^2 + k_a)|_{s=j\omega} = 0$$
$$\omega = \sqrt{\frac{k_a}{m_a}}$$

so,

$$k_a = \omega^2 m_a = 10^2 \cdot 5 = 500\, N/m$$