2.016 Hydrodynamics

Prof. A.H. Techet

Fluid Forces on Bodies

1. Steady Flow

In order to design offshore structures, surface vessels and underwater vehicles, an understanding of the basic fluid forces acting on a body is needed. In the case of steady viscous flow, these forces are straightforward. *Lift* force, perpendicular to the velocity, and *Drag* force, inline with the flow, can be calculated based on the fluid velocity, U, force coefficients, C_D and C_L, the object's dimensions or area, A, and fluid density, ρ. For viscous flows the drag and lift on a body are defined as follows

$$F_{Drag} = \frac{1}{2}\rho U^2 A\, C_D \tag{5.1}$$

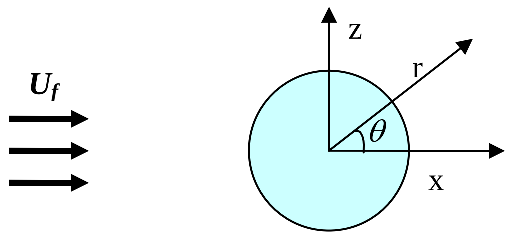
$$F_{Lift} = \frac{1}{2}\rho U^2 A\, C_L \tag{5.2}$$

These equations can also be used in a quiescent (stationary) fluid for a steady translating body, where U is the body velocity instead of the fluid velocity, since U is still the relative velocity of the fluid with respect to the body.

The drag force arises due to viscous rubbing of the fluid. The fluid may be thought of as comprised of several "layers" which move relative to one another. The layer at the surface of the body "sticks" to the surface due to the *no-slip condition*. The next layer of fluid away from the surface rubs against the layer below, and this rubbing requires a certain amount of force because of viscosity. One would expect that in the absence of viscosity, the force would go to zero.

Jean Le Rond d'Alembert (1717-1783) performed a series of experiments to measure the drag on a sphere in a flowing fluid, and on the basis of the potential flow analysis he expected that the force would approach zero as the viscosity of the fluid approached zero. However, this was not the case. The net force seemed to converge on a non-zero value as the viscosity approached zero. Hence, the vanishing of the net force in the potential flow analysis is known as d'Alembert's Paradox.

D'Allembert's Paradox for a fixed sphere in uniform inflow: Force on a sphere (radius *a*) in an unbounded STEADY moving fluid with velocity U_f is explored in the following discussion.



The corresponding 3D potential function for a sphere in uniform inflow is simply:

$$\phi(r,\theta) = U_f \left(r + \frac{a^3}{2r^2} \right) \cos\theta \tag{5.3}$$

The hydrodynamic force on the body due to the unsteady motion of the sphere is given as a surface integral of pressure around the body. Pressure formulation comes from the unsteady form of Bernoulli. Force in the x-direction is

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS \tag{5.4}$$

Here, the time derivative of the potential is zero since the flow is steady and velocity is not a function of time. Since we want the force acting on the body we need the velocity components on the sphere surface ($r = a$). In spherical coordinates the velocity is found by taking the gradient of the potential function as follows:

$$\vec{V} = \nabla \phi = \left(V_r, V_\theta, V_\varphi \right) = \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \varphi} \right) \tag{5.5}$$

The velocity at the body, on *r* = *a*, can only be tangential to the body due to the kinematic boundary condition (KBC $\Rightarrow$ $\mathrm{V_r} = 0\big|_{r=a}$):

$$\vec{V} = \nabla \phi\big|_{r=a} = \left(0, -\tfrac{3}{2} U_f \sin\theta, 0 \right) \tag{5.6}$$

The magnitude of the velocity is simply:

$$\frac{1}{2}\left|\nabla^2\phi\right| = \frac{9}{8}U^2 \sin^2\theta \qquad (5.7)$$

Using (5.7) in the formulation for F_x, from equation (5.4), we get the horizontal force on the body:

$$F_x = (-\rho)\left(2\pi a^2\right)\int_0^\pi -\cos\theta\sin\theta\left[\tfrac{9}{8}U_f^2\sin^2\theta\right]d\theta \qquad (5.8)$$

$$F_x = \tfrac{9}{4}\rho\pi a^2 U_f^2 \int_0^\pi \sin^3\theta\cos\theta\, d\theta \qquad (5.9)$$

This integral can be integrated by parts or substitution of variables:

$$p = \sin\theta;\ dp = \cos\theta\, d\theta \qquad (5.10)$$

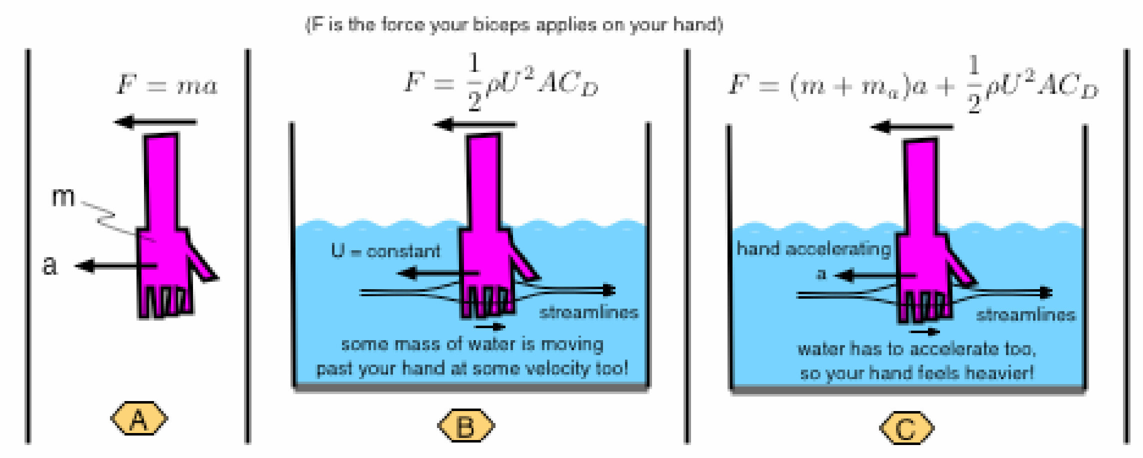
$$\int \sin^3\theta\cos\theta\, d\theta = \int p^3 dp = \frac{p^4}{4} = \frac{\sin^4\theta}{4} \qquad (5.11)$$

$$\therefore F_x = \tfrac{9}{4}\rho\pi a^2 U_f^2\left[\frac{\sin^4\theta}{4}\right]_0^\pi = 0 \qquad (5.12)$$

THERE IS NO FORCE ON A BODY IN A STEADY FLOW IN THE ABSENCE OF VISCOSITY! THIS IS D'ALLEMBERT'S PARADOX.

The resolution becomes clear when we realize that any non-zero viscosity, no matter how small, will result in a boundary layer and the tangential flow velocity vanishing at the surface of the sphere. As we lower the viscosity, the thickness of the boundary layer is reduced, but the flow velocity still drops to zero across that layer (the "no-slip" condition). The results of this boundary layer lead to losses in the momentum of the flowing fluid and the transference of momentum to the sphere, i.e., to a net unbalanced force.

2. Unsteady Motion and Added Mass.



Do this at home:

A) Wave your hand in the air. Feel the force it takes to accelerate your hand. F=ma.

B) Fill your bathtub with water. Run your hand through (with the palm facing forward) at a slow, constant speed. Feel the drag on your hand. Notice that the water must move to flow around your hand.

B1.5) Run your hand through at another constant, faster speed. Notice that it takes more force. Recall the drag force is proportional to U^2. Notice that the water now moves at some constant, faster speed around your hand.

C) Try and accelerate your hand from the slow speed to the fast speed. It's hard, huh?! Notice that the water flowing past your hand has to accelerate as your hand accelerates. Since some mass of water must accelerate, your hand feels heavier. We capture this idea with the concept of ***added mass***.

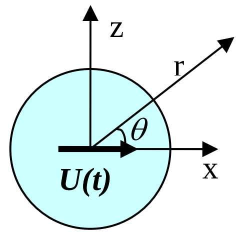
Beyond steady flow, especially in the presence of free surface waves, we must consider unsteady, time dependent motions of both the fluid and the body and the fluid inertial forces arise, adding to the total forcing on a body.

Take the case of an unsteady moving body, $U_b(t)$, in an unbounded inviscid, irrotational fluid ($\mu = 0$) with zero velocity, $U_f = 0$. The time-dependent force on the body is directly proportional to the body acceleration:

$$F(t) = -m_a \frac{dU_b(t)}{dt} \qquad (5.13)$$

where m_a, is the system added mass, depends on the body geometry and direction of motion. This is an *added* inertial force or *added mass* force on the body. By comparison, in an inviscid steady flow, by D'Alembert's Paradox, the force on the body would be zero.

Unsteady Moving Body Stationary Fluid: Force on a sphere (radius *a*) accelerating in an unbounded quiescent (non-moving) fluid. $U = U(t)$ is the unsteady body velocity.



The Kinematic Boundary Condition on the sphere, guaranteeing no fluid flow through the body surface, is

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = U(t)\cos\theta. \qquad (5.14)$$

The potential function for a moving sphere with no free stream (still fluid) is simply

$$\phi = -U(t)\frac{a^3}{2r^2}\cos\theta. \qquad (5.15)$$

You can double check this solution for the velocity potential by substituting ϕ into the Kinematic Boundary Condtion (eq. (5.14)) to make sure this potential works at the boundary of the sphere.

The hydrodynamic force on the body due to the unsteady motion of the sphere is given as a surface integral of pressure around the body. Pressure formulation comes from the unsteady form of Bernoulli. Force in the x-direction is

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS \qquad (5.16)$$

Since we want the force acting on the body we need the velocity components on the sphere surface ($r = a$).

In spherical coordinates the velocity is found by taking the gradient of the potential function as follows:

$$\vec{V} = \nabla\phi = \left(V_r, V_\theta, V_\varphi\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) \tag{5.17}$$

such that the gradient of $\phi(t)$ for a moving sphere is

$$\nabla\phi\big|_{r=a} = \left(U(t)\cos\theta, \frac{1}{2}U(t)\sin\theta, 0\right). \tag{5.18}$$

To evaluate the pressure around the sphere we need the magnitude of the velocity $\left|\nabla\phi\right|^2$ on the surface ($r = a$):

$$\left(\left|\nabla\phi\right|^2\right)\Big|_{r=a} = U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta;\;\; \hat{n} = -\hat{e}_r;\;\; n_x = -\cos\theta \tag{5.19}$$

Next, the time derivative of the velocity potential, evaluated at the sphere surface, is

$$\frac{\partial\phi}{\partial t}\bigg|_{r=a} = -\dot{U}(t)\frac{a^3}{2r^2}\cos\theta\Big|_{r=a} = -\frac{1}{2}\dot{U}(t)\,a\cos\theta \tag{5.20}$$

and the surface integral can be re-written in spherical coordinates as

$$\iint_b dS = \int_0^\pi (a\,d\theta)(2\pi a\sin\theta) \tag{5.21}$$

Substituting (5.19), (5.20), and (5.21) into (5.16) we can solve for *Fx*, the added mass force on a spherical body moving with an unsteady acceleration:

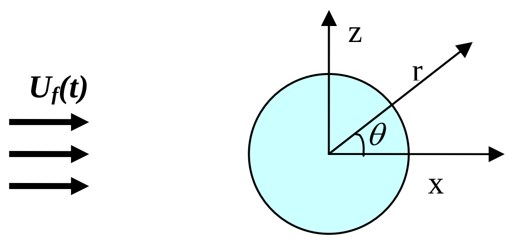
$$F_x = -\dot{U}(t)\left(\underbrace{\rho\frac{2}{3}\pi a^3}_{unit=Mass}\right) \tag{5.22}$$

The volume of a sphere is $\forall_s = \frac{4}{3}\pi a^3$ thus equation (5.22) is simply

$$\boxed{F_x = -\dot{U}(t)\left(\tfrac{1}{2}\rho\forall_s\right)} \tag{5.23}$$

where $m_a = \frac{1}{2}\rho\forall_s$ is the added mass in the system.

Unsteady Moving Fluid Stationary Body: Force on a sphere (radius *a*) in an unbounded unsteady moving fluid. $U_f = U_f(t)$ is the unsteady fluid velocity.



The corresponding potential function is simply:

$$\phi(r,\theta,t) = U_f(t)\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad (5.24)$$

The velocity, at the body on r=a, can only be tangential to the body due to the kinematic boundary condition ($\text{KBC} \Rightarrow \text{V}_\text{r} = 0\big|_{r=a}$) :

$$\vec{V} = \nabla\phi\big|_{r=a} = \left(0, -\tfrac{3}{2}U_f \sin\theta, 0\right) \qquad (5.25)$$

The time derivative of the potential is

$$\left.\frac{\partial\phi}{\partial t}\right|_{r=a} = \dot{U}_f \frac{3a}{2}\cos\theta \qquad (5.26)$$

The magnitude of the velocity is simply:

$$\frac{1}{2}\left|\nabla^2\phi\right| = \frac{9}{8}U^2 \sin^2\theta \qquad (5.27)$$

Again, using (5.26), (5.27), (5.21) in the formulation for F_x, from equation (5.16), we get the horizontal force on the body:

$$F_x = (-\rho)\left(2\pi a^2\right)\int_0^{\pi} -\cos\theta\sin\theta\left[\dot{U}_f\,\tfrac{3a}{2}\cos\theta + \tfrac{9}{8}U_f^2\sin^2\theta\right]d\theta \tag{5.28}$$

$$F_x = 3\rho\pi a^3\,\dot{U}_f\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta + \tfrac{9}{4}\rho\pi a^2\,U_f^2\int_0^{\pi}\sin^3\theta\cos\theta\,d\theta \tag{5.29}$$

By parts the right most term in the integral in (5.29) reduces to zero, and the integral from left term is simply:

$$\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta = \frac{2}{3} \tag{5.30}$$

so we are left with a force:

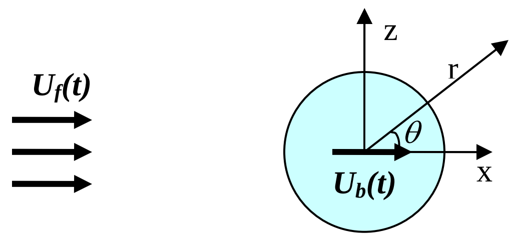
$$F_x = 3\rho\pi a^3\,\dot{U}_f\cdot\frac{2}{3} = 2\rho\pi a^3\,\dot{U}_f \tag{5.31}$$

This can be rewritten in terms of the sphere volume, $\forall_s = \frac{4}{3}\pi a^3$, and added mass, $m_a = \frac{1}{2}\rho\forall_s$, from (5.23) as follows

$$\boxed{F_x = \dot{U}_f\left(\rho\forall + m_a\right)} \tag{5.32}$$

The non-added mass term of equation (5.32) is due to the pressure gradient necessary to accelerate the fluid around the sphere. This is like a buoyancy effect.

Unsteady Moving Fluid; Unsteady Moving Body: Force on a moving sphere (radius *a*) in an unbounded moving fluid. $U_f = U_f(t)$ is the unsteady fluid velocity and $U_b = U_b(t)$ is the body velocity.



The case of the unsteady moving body and fluid can be determined by combining the results from the previous two cases.

$$\boxed{F_x = -\dot{U}_b\left(\tfrac{1}{2}\rho\forall_s\right)}$$

Moving Body
Still Fluid

$$\boxed{F_x = \dot{U}_f\left(\rho\forall + m_a\right)}$$

Moving Fluid
Still Body

Moving Body, Moving Fluid:

$$F_x = -\dot{U}_b\left(m_a\right) + \dot{U}_f\left(\rho\forall + m_a\right) = \dot{U}_f\rho\forall + m_a\left(\dot{U}_f - \dot{U}_b\right) \quad (5.33)$$

So now, all we have to do is find the added mass!

