2.016 Hydrodynamics

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Added Mass

For the case of unsteady motion of bodies underwater or unsteady flow around objects, we must consider the additional effect (force) resulting from the fluid acting on the structure when formulating the system equation of motion. This added effect is *added mass.* Most floating structures can be modeled, for small motions and linear behavior, by a system equation with the basic form similar to a typical mass-spring-dashpot system described by the following equation:

$$m\ddot{x} + b\dot{x} + kx = f(t) \tag{6.1}$$

where m is the system mass, b is the linear damping coefficient, k is the spring coefficient, $f(t)$ is the force acting on the mass, and x is the displacement of the mass. The natural frequency ω of the system is simply

$$\omega = \sqrt{\frac{k}{m}}\,. \tag{6.2}$$

In a physical sense, this added mass is the weight added to a system due to the fact that an accelerating or decelerating body (ie. unsteady motion: $dU/dt \neq 0$) must move some volume of surrounding fluid with it as it moves. The added mass force opposes the motion and can be factored into the system equation as follows:

$$m\ddot{x} + b\dot{x} + kx = f(t) - m_a\ddot{x} \tag{6.3}$$

where m_a is the added mass. Reordering the terms the system equation becomes:

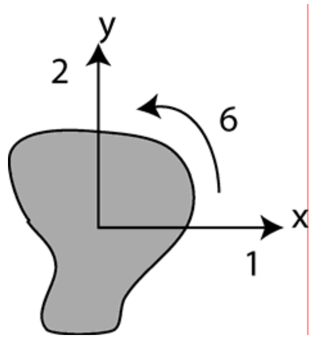
$$\left(m + m_a\right)\ddot{x} + b\dot{x} + kx = f(t) \tag{6.4}$$

From here we can treat this again as a simple spring-mass-dashpot system with a new mass $m' = m + m_a$ such that the natural frequency of the system is now

$$\omega' = \sqrt{\frac{k}{m'}} = \sqrt{\frac{k}{m + m_a}} \tag{6.5}$$

It is important in ocean engineering to consider floating vessels or platforms motions in more than one direction. Added mass forces can arise in one direction due to motion in a different direction, and thus we can end up with a 6 x 6 matrix of added mass coefficients.

Looking simply at a body in two-dimensions we can have linear motion in two directions and rotational motion in one direction. (Think of these coordinates as if you were looking down on a ship.)



Two dimensional motion with axis (x,y) fixed on the body. 1: Surge, 2: Sway, 6: Yaw

The unsteady forces on the body in the three directions are:

$$-F_1 = m_{11}\frac{du_1}{dt} + m_{12}\frac{du_2}{dt} + m_{16}\frac{du_6}{dt} \tag{6.6}$$

$$-F_2 = m_{21}\frac{du_1}{dt} + m_{22}\frac{du_2}{dt} + m_{26}\frac{du_6}{dt} \tag{6.7}$$

$$-F_6 = m_{61}\frac{du_1}{dt} + m_{62}\frac{du_2}{dt} + m_{66}\frac{du_6}{dt} \tag{6.8}$$

Where F_1, F_2, and F_6, are the surge (x-) force, sway (y-) force and yaw moments respectively. It is common practice in Ocean Engineering and Naval Architecture to write the moments for roll, pitch, and yaw as F_4, F_5, and F_6 and the angular motions in these directions as X_4, X_5, and X_6.

This set of equations, (6.6)-(6.8), can be written in matrix form, $\underline{F} = [M]\dot{\underline{u}}$,

$$\underline{F} = \begin{bmatrix} m_{11} & m_{12} & m_{16} \\ m_{21} & m_{22} & m_{26} \\ m_{61} & m_{62} & m_{66} \end{bmatrix} \begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \\ \frac{du_6}{dt} \end{pmatrix} \tag{6.9}$$

Considering all six degrees of freedom the Force Matrix is:

$$\underline{F} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{bmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix} \tag{6.10}$$

We will often abbreviate how we write the Force matrix given in (6.10) using tensor notation.

The force vector is written as

$$\underline{F} = F_i \text{, where } i = \underbrace{1,2,3}_{\substack{\text{Linear} \\ \text{Forces}}}, \underbrace{4,5,6}_{\text{Moments}}, \tag{6.11}$$

the acceleration vector as

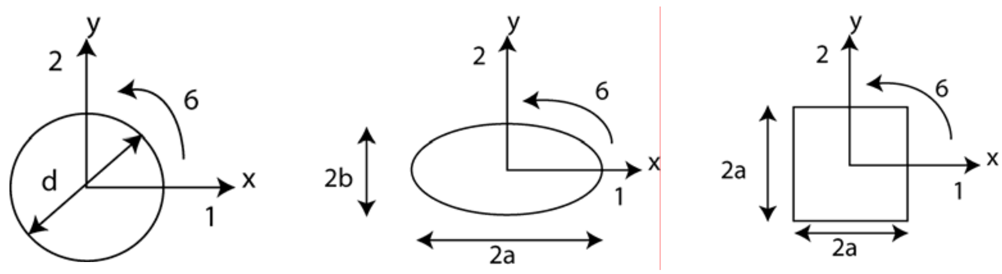
$$\dot{u}_i = [u_1, u_2, u_3, u_4, u_5, u_6], \tag{6.12}$$

and the added mass matrix $[m_a]$ as

$$m_{ij} \text{ where } i, j = 1,2,3,4,5,6. \tag{6.13}$$

A good way to think of the added mass components, m_{ij}, is to think of each term as mass associated with a force on the body in the i^{th} direction due to a *unit* acceleration in the j^{th} direction.

For symmetric geometries the added mass tensor simplifies significantly. For example, figure 2 shows added mass values for a circle, ellipse, and square. In the case of the circle and square, movement in the 1 and 2 directions yields similar geometry and identical added mass coefficients ($m_{11} = m_{22}$).



Circle
$m_{11} = m_{22} = \rho d^2$
$m_{66} = 0$

Ellipse
$m_{11} = \rho \pi b^2$
$m_{22} = \rho \pi a^2$
$m_{66} = \rho \left(a^2 - b^2 \right)^2$

Square
$m_{11} = m_{22} = 1.51 \pi \rho a^2$
$m_{66} = 0.234 \pi \rho a^4$

Two dimensional added mass coefficients for a circle, ellipse, and square in 1: Surge, 2: Sway, 6: Yaw

Using these coefficients and those tabulated in Newman's *Marine Hydrodynamics* on p.145 we can determine the added mass forces quite simply.

In three-dimensions, for a sphere (by symmetry):

$$m_{11} = m_{22} = m_{33} = \frac{1}{2} \rho \forall = m_A \tag{6.14}$$

ALL OTHER m_{ij} TERMS ARE ZERO ($i \neq j$).

General 6 DOF forces and moments on a Rigid body moving in a fluid:

Velocities:

$$\textit{Translation Velocity}: \vec{U}(t) = (U_1, U_2, U_3) \tag{6.15}$$

$$\textit{Rotational Velocity}: \vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6) \tag{6.16}$$

All rotation is taken with respect to Origin of the coordinate system (often placed at the center of gravity of the object for simplicity!).

Forces: (force in the j^{th} direction). ($i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$)

$$F_j = -\dot{U}_i m_{ij} - \varepsilon_{jkl} U_i \Omega_k m_{li} \tag{6.17}$$

Moments: ($i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$)

$$M_j = -\dot{U}_i m_{j+3,\, i} - \varepsilon_{jkl} U_i \Omega_k m_{l+3,\, i} - \varepsilon_{jkl} U_k U_i m_{li} \tag{6.18}$$

Einstein's summation notation applies!

The alternating tensor ε_{jkl} is simply

$$\varepsilon_{jkl} = \begin{cases} 0; & \text{if any } j,k,l \text{ are equal} \\ 1; & \text{if } j,k,l \text{ are in cyclic order} \\ -1; & \text{if } j,k,l \text{ are in anti-cyclic order} \end{cases} \tag{6.19}$$

The full form of the force in the x-direction (*F_1*) is summed over all values of *i*:

$$\begin{aligned} \underbrace{F_1}_{j=1} = & -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6} \\ & - \underbrace{\varepsilon_{1kl} U_1 \Omega_k m_{l1}}_{i=1} - \underbrace{\varepsilon_{1kl} U_2 \Omega_k m_{l2}}_{i=2} - \underbrace{\varepsilon_{1kl} U_3 \Omega_k m_{l3}}_{i=3} - \underbrace{\varepsilon_{1kl} U_4 \Omega_k m_{l4}}_{i=4} \\ & - \underbrace{\varepsilon_{1kl} U_5 \Omega_k m_{l5}}_{i=5} - \underbrace{\varepsilon_{1kl} U_6 \Omega_k m_{l6}}_{i=6} \end{aligned} \tag{6.20}$$

for $k, l = 1, 2, 3$.

Next we can choose the index *k* to cycle through. It is helpful to note that the only terms where *k* plays a role, contain ε_{jkl}. Following the definition for ε_{jkl} given in (6.19) and since *j = 1*, all terms will be zero for *k = 1*. Therefore *k* can only take the value of 2 or 3:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

$$\underbrace{-\underbrace{\varepsilon_{12l} U_1 \Omega_2 m_{l1}}_{i=1} - \underbrace{\varepsilon_{12l} U_2 \Omega_2 m_{l2}}_{i=2} - \underbrace{\varepsilon_{12l} U_3 \Omega_2 m_{l3}}_{i=3} - \underbrace{\varepsilon_{12l} U_4 \Omega_2 m_{l4}}_{i=4} - \underbrace{\varepsilon_{12l} U_5 \Omega_2 m_{l5}}_{i=5} - \underbrace{\varepsilon_{12l} U_6 \Omega_2 m_{l6}}_{i=6}}_{k=2}$$

$$\underbrace{-\underbrace{\varepsilon_{13l} U_1 \Omega_3 m_{l1}}_{i=1} - \underbrace{\varepsilon_{13l} U_2 \Omega_3 m_{l2}}_{i=2} - \underbrace{\varepsilon_{13l} U_3 \Omega_3 m_{l3}}_{i=3} - \underbrace{\varepsilon_{13l} U_4 \Omega_3 m_{l4}}_{i=4} - \underbrace{\varepsilon_{13l} U_5 \Omega_3 m_{l5}}_{i=5} - \underbrace{\varepsilon_{13l} U_6 \Omega 3 m_{l6}}_{i=6}}_{k=3}$$

(6.21)

Finally we cycle through the index *l*. Again it is helpful to note that the only terms where *l* plays a role, contain ε_{jkl}. Following the definition for ε_{jkl} given in (6.19) and since *j = 1*, and *k = 2* or *3*, then all terms will be zero for *l = 1* and some zero for the case *l = 2* and others zero when *l =3*. Like before *l* can only take the value of 2 or 3 such that $l \neq k \neq j$:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

$$\underbrace{-\underbrace{\varepsilon_{123} U_1 \Omega_2 m_{31}}_{i=1} - \underbrace{\varepsilon_{123} U_2 \Omega_2 m_{32}}_{i=2} - \underbrace{\varepsilon_{123} U_3 \Omega_2 m_{33}}_{i=3} - \underbrace{\varepsilon_{123} U_4 \Omega_2 m_{34}}_{i=4} - \underbrace{\varepsilon_{123} U_5 \Omega_2 m_{35}}_{i=5} - \underbrace{\varepsilon_{123} U_6 \Omega_2 m_{36}}_{i=6}}_{k=2;\, l=3}$$

$$\underbrace{-\underbrace{\varepsilon_{132} U_1 \Omega_3 m_{21}}_{i=1} - \underbrace{\varepsilon_{132} U_2 \Omega_3 m_{22}}_{i=2} - \underbrace{\varepsilon_{132} U_3 \Omega_3 m_{23}}_{i=3} - \underbrace{\varepsilon_{132} U_4 \Omega_3 m_{24}}_{i=4} - \underbrace{\varepsilon_{132} U_5 \Omega_3 m_{25}}_{i=5} - \underbrace{\varepsilon_{132} U_6 \Omega 3 m_{26}}_{i=6}}_{k=3;\, l=2}$$

(6.22)

On the second row of the equation above, the indices of the alternating tensor, ε_{jkl}, are in cyclic order *jkl = 123* ($\varepsilon_{123} = +1$). In the third row, the indices are in anti (or reverse) cyclic order: $\varepsilon_{132} = -1$ where *jkl = 132.*

More than likely you will never have to write out all six force equations with all the terms as the velocity and acceleration of the body will be zero in certain directions. However for a full seakeeping analysis of a ship then one day you just might need to be able to determine all the forces!

Typical Example: For a body moving in the fluid with velocity

$$\vec{V} = (1,0,1,0,0,1) = (U_1, 0, U_3, 0, 0, U_6) = (U_1, 0, U_3, 0, 0, \Omega_3) \tag{6.23}$$

and acceleration

$$\vec{a} = (1,0,0,0,0,1) = (\dot{U}_1, 0, 0, 0, 0, \dot{U}_6) \tag{6.24}$$

we can find the force on the body in the X-direction. The force in the x-direction is F_1 so $j=1$.

First substitute "1" for every instance of *j* in equation (6.17) to get:

$$F_{j=1} = F_1 = -\dot{U}_i\, m_{i1} - \varepsilon_{1kl}\, U_i \Omega_k\, m_{li} \tag{6.25}$$

Next we need to "cycle" through the possible values for *i* (*i = 1,2,3,4,5,6*). Looking at equation (6.25), it is clear that the only "i^{th}" accelerations that will matter are the non-zero ones from (6.24), thus $\dot{U}_1$ and $\dot{U}_6$, and the only "i^{th}" velocities to consider are for *i = 1,3,* and *6* [eqn (6.23)].

$$F_1 = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_6 m_{61}}_{i=6} - \underbrace{\varepsilon_{1kl}\, U_1 \Omega_k\, m_{l1}}_{i=1} - \underbrace{\varepsilon_{1kl}\, U_3 \Omega_k\, m_{l3}}_{i=3} - \underbrace{\varepsilon_{1kl}\, U_6 \Omega_k\, m_{l6}}_{i=6} \tag{6.26}$$

Now look at the *k-index*: $(k \neq j \therefore k = 2,3)$ However $\Omega_2 = 0$ and $\Omega_3 \neq 0$ thus for $k = 2$ all associated terms will be zero, so we only have to deal with $k = 3$. Since $j = 1$ and $k = 3$ the only value left for *l*, that could result in non-zero terms, is 2.

$$\begin{aligned} F_1 = &-\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_6 m_{61}}_{i=6} \\ &\underbrace{-\underbrace{\varepsilon_{132}\, U_1 \Omega_3\, m_{21}}_{i=1} - \underbrace{\varepsilon_{132}\, U_3 \Omega_3\, m_{23}}_{i=3} - \underbrace{\varepsilon_{132}\, U_6 \Omega_3\, m_{26}}_{i=6}}_{k=3;\, l=2} \end{aligned} \tag{6.27}$$

If the body in question was a simple, symmetrical sphere we could reduce this even further. Using the added mass values from (6.14) and trusting that the off-diagonal added mass terms are zero (just for the sphere), the force in the x-direction on a sphere, given (6.23) and (6.24), is

$$\boxed{F_1 = -\underbrace{\dot{U}_1 m_{11}}_{i=1}} \tag{6.28}$$

Determining 3D Added Mass Using Slender Body Theory

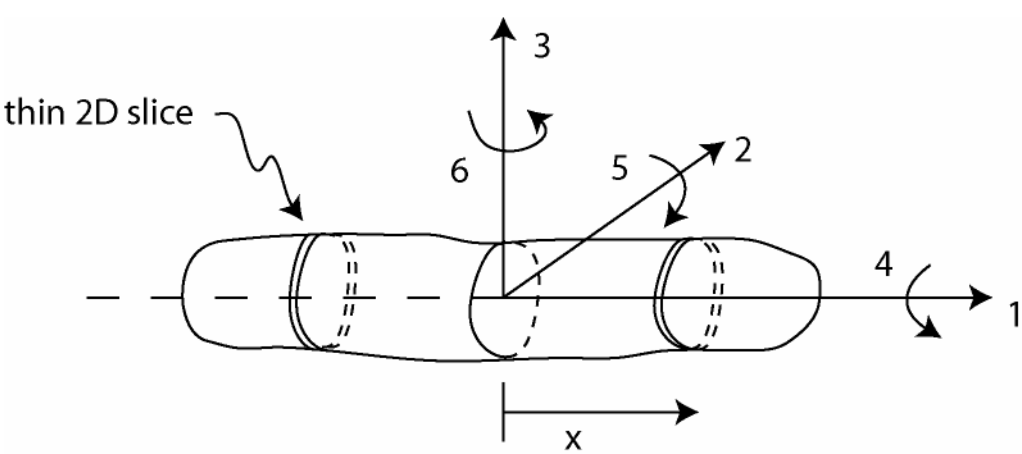
To formulate the added mass of a system such as a ship or submarine that can be modeled as a *slender body*, we first need the two-dimensional sectional added mass coefficients. We will consider a *slender body* to have a characteristic length in one direction that is considerably longer than its length in the other two directions. For these slender bodies we can use known 2D coefficients to find the unknown 3D added mass coefficient for the body.

The added mass *force* acting on the body due to unsteady motion is

$$F_j = -\dot{U}_i m_{ij} - \varepsilon_{jkl} U_i \Omega_k m_{li} \tag{6.29}$$

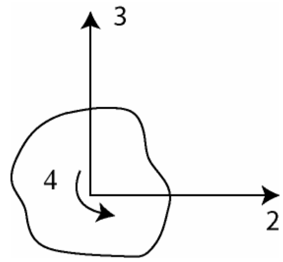
where m_{ij} is the added mass in the i^{th} direction due to a unit acceleration in the j^{th} direction and $i,j = 1{:}6$. The added mass tensor, m_{ij}, is symmetric!

To find the 3D added mass coefficients consider simply the body geometry, ignoring for now the actual motions of the vessel. To start, orient the 1-axis along the long axis of the slender body as shown in figure 1. The 3D added mass coefficients will be found by summing (or integrating) the added mass coefficients of the 2D cross-sectional slices along the body.



Slender body oriented with the long axis in the 1-direction.

The sectional added mass coefficients are tabulated for simple geometries. In general, with the slender body aligned lengthwise along the 1-axis, the 2D cross-sectional slice is aligned with the 2-3 plane, some distance *x* from the origin (figure 1). This 2D slice is shown in figure 2. To find the 3D coefficients we need to know the 2D coefficient of each section (strip) along the length of the vessel. For a uniform diameter cylinder this is quite simple, but for ships with complex geometry there is a bit more work involved.



2D cross-sectional slice of slender body.

The 2D coefficients will be written as a_{ij} whereas the 3D coefficients are written as m_{ij}. From here on we will follow the basic formulations used in the handbook: Principles of Naval Architecture Vol III., (1989) Soc. Naval Arch. and Marine Engineers, p. 56.

	1	2	3	4	5	6
1						
2		$m_{22} = \int_L a_{22}\, dx$	$m_{23} = -\int_L a_{23}\, dx$	$m_{24} = \int_L a_{24}\, dx$		$m_{26} = \int_L x\, a_{22}\, dx$
3			$m_{33} = \int_L a_{33}\, dx$		$m_{35} = -\int_L x\, a_{33}\, dx$	
4				$m_{44} = \int_L a_{44}\, dx$		$m_{46} = \int_L x\, a_{24}\, dx$
5					$m_{55} = \int_L x^2\, a_{33}\, dx$	
6						$m_{66} = \int_L x^2\, a_{22}\, dx$

