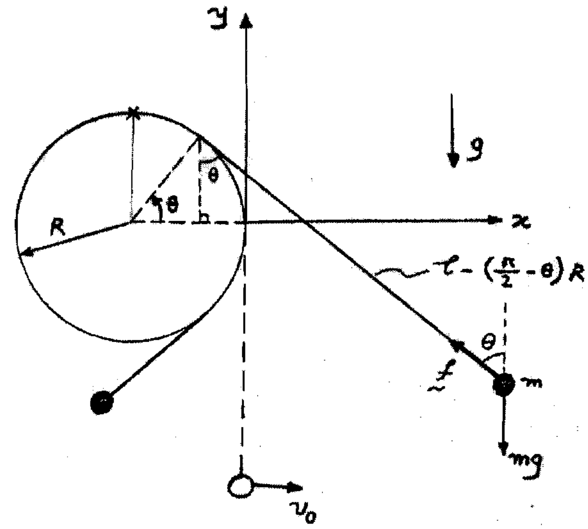
Problem Set No. 1

Problem 1



$$\begin{cases} x_m = \left[\ell-\left(\frac{\pi}{2}-\theta\right)R\right]\sin\theta - R(1-\cos\theta) \\ y_m = -\left[\ell-\left(\frac{\pi}{2}-\theta\right)R\right]\cos\theta + R\sin\theta \end{cases}$$

$$\begin{cases} \dot{x}_m = \left[\ell-\left(\frac{\pi}{2}-\theta\right)R\right]\cos\theta.\dot{\theta} \\ \dot{y}_m = \left[\ell-\left(\frac{\pi}{2}-\theta\right)R\right]\sin\theta.\dot{\theta} \end{cases}$$

$$\underset{\sim}{v}_m = \dot{x}_m \hat{\underset{\sim}{e}}_x + \dot{y}_m \hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{f} = |\underset{\sim}{f}|\left(-\sin\theta\,\hat{\underset{\sim}{e}}_x + \cos\theta\,\hat{\underset{\sim}{e}}_y\right)$$

$\therefore \quad \underset{\sim}{f}\cdot\underset{\sim}{v}_m = 0$

$\Longrightarrow$ $\underset{\sim}{f}$ does not work $\Longrightarrow$ The only force that does work is gravitational force which is conservative. $\Longrightarrow$ System is conservative

$\Longrightarrow$ $KE + PE = \text{const.}$

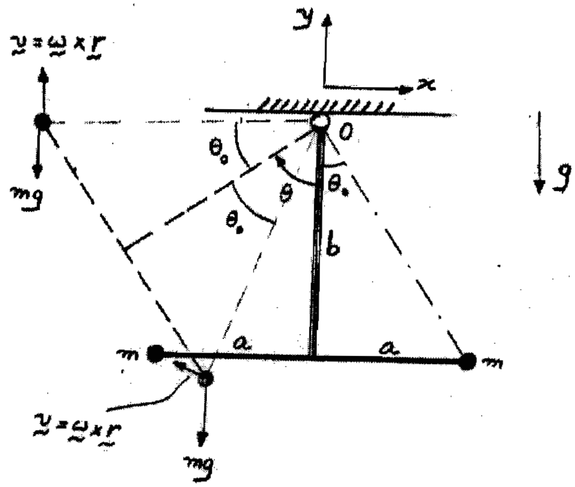
At $\theta = 0$, $\begin{cases} KE = \frac{1}{2}m v_m^2 = \frac{1}{2}m v_0^2 \\ PE = mg\,y_m(\theta=0) = -mg\left(\ell-\frac{\pi}{2}R\right) \end{cases}$

At two extreme deflections, $v_m = 0 \Longrightarrow KE = 0$

$$\therefore \quad PE\Big|_{\theta=\theta_{max/min}} = (PE+KE)\Big|_{\theta=0} = \frac{1}{2}m v_0^2 - mg\left(\ell-\frac{\pi}{2}R\right) = mg\,y_m\Big|_{\theta=\theta_{max/min}}$$

$$\Longrightarrow \quad y_m\Big|_{\theta=\theta_{max/min}} = \frac{v_0^2}{2g} - \ell + \frac{\pi}{2}R = \left[R\sin\theta - \left[\ell-\left(\frac{\pi}{2}-\theta\right)R\right]\cos\theta\right]_{\theta=\theta_{max}/\theta_{min}}$$

Problem 2



$$\theta_0 = \tan^{-1}\left(\frac{a}{b}\right)$$

$$|\underset{\sim}{r}| = \sqrt{a^2+b^2}$$

$$|\underset{\sim}{\omega}| = \dot{\theta}$$

$$\underset{\sim}{v}_0 = 0 \quad \Longrightarrow \quad \underset{\sim}{\tau}_0^{ext} = \frac{d\underset{\sim}{h}_0}{dt}$$

$$\underset{\sim}{\tau}_0 = \left\{ mg\sqrt{a^2+b^2}\ \sin(\theta+\theta_0) + mg\sqrt{a^2+b^2}\ \sin(\theta-\theta_0) \right\} \hat{\underset{\sim}{e}}_z$$

$$= mg\sqrt{a^2+b^2}\ (2\sin\theta\cos\theta_0)\ \hat{\underset{\sim}{e}}_z$$

$$|\underset{\sim}{v}| = \sqrt{a^2+b^2}\ \dot{\theta} \quad \longrightarrow \quad |\underset{\sim}{P}| = m\sqrt{a^2+b^2}\ \dot{\theta}$$

$$\underset{\sim}{h}_0 = -2\sqrt{a^2+b^2}\left(m\sqrt{a^2+b^2}\ \dot{\theta}\right)\hat{\underset{\sim}{e}}_z$$

Angular momentum about point O :

$$2mg\sqrt{a^2+b^2}\ \sin\theta\ \cos\theta_0 = -2m(a^2+b^2)\ \ddot{\theta}$$

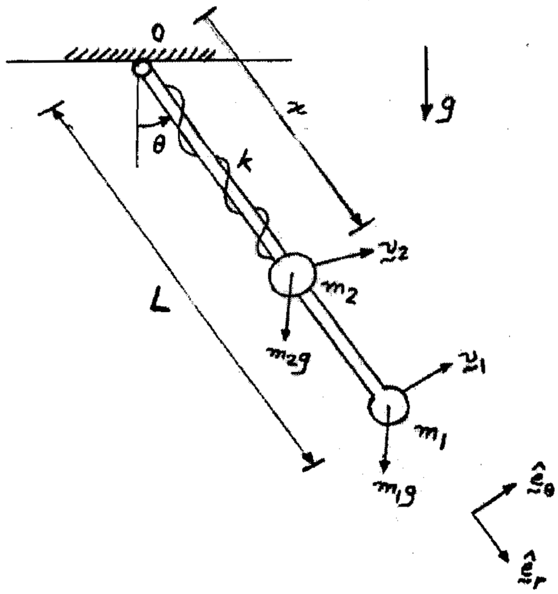
$$\cos\theta_0 = \frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore \quad \ddot{\theta} + \frac{gb}{a^2+b^2}\sin\theta = 0$$

equation of motion for $\theta(t)$

Problem 3

Assume free length of spring to be L_0.



a) One needs two coordinates θ and x to describe the motion of the system.

θ and x are a complete and independent set of generalized coordinates.

b) We need to find two equations:

First, angular momentum about point O for the system:

$$\underset{\sim}{\tau}_0^{ext} = \frac{d\underset{\sim}{h}_0}{dt}$$

$$\underset{\sim}{\tau}_0 = -\left(m_1 g L \sin\theta + m_2 g x \sin\theta\right) \underset{\sim}{\hat{e}}_z$$

$$\underset{\sim}{v}_1 = L\dot{\theta}\,\underset{\sim}{\hat{e}}_\theta \qquad \& \qquad \underset{\sim}{v}_2 = \dot{x}\,\underset{\sim}{\hat{e}}_r + x\dot{\theta}\,\underset{\sim}{\hat{e}}_\theta$$

$$\underset{\sim}{h}_0 = \sum_{i=1}^{2} \underset{\sim}{r}_i \times \underset{\sim}{P}_i = (L\,\underset{\sim}{\hat{e}}_r)\times(m_1 L\dot{\theta}\,\underset{\sim}{\hat{e}}_\theta) + (x\,\underset{\sim}{\hat{e}}_r)\times\left[m_2(\dot{x}\,\underset{\sim}{\hat{e}}_r + x\dot{\theta}\,\underset{\sim}{\hat{e}}_\theta)\right]$$

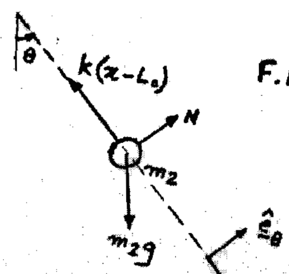
$$\Longrightarrow \underset{\sim}{h}_0 = \left(m_1 L^2\dot{\theta} + m_2 x^2\dot{\theta}\right)\underset{\sim}{\hat{e}}_z$$

$$\therefore \quad -\left(m_1 g L \sin\theta + m_2 g x \sin\theta\right) = \left(m_1 L^2 + m_2 x^2\right)\ddot{\theta} + 2m_2 x\dot{x}\dot{\theta}$$

$$\Longrightarrow \quad \left(m_1 L^2 + m_2 x^2\right)\ddot{\theta} + 2m_2 x\dot{x}\dot{\theta} + \left(m_1 L + m_2 x\right) g \sin\theta = 0$$

To find the second equation, apply linear momentum in the radial direction for m_2:

m_2 is free to slide along the rod so the force N has no component along the radial direction.



F.B.D.

Problem 3

$$\mathbf{v}_2 = \dot{x}\,\hat{\mathbf{e}}_r + x\dot{\theta}\,\hat{\mathbf{e}}_\theta$$

$$\mathbf{a}_2 = \frac{d\mathbf{v}_2}{dt} = \ddot{x}\,\hat{\mathbf{e}}_r + \dot{x}\underbrace{\frac{d\hat{\mathbf{e}}_r}{dt}}_{\dot{\theta}\,\hat{\mathbf{e}}_\theta} + (x\ddot{\theta} + \dot{x}\dot{\theta})\,\hat{\mathbf{e}}_\theta + x\dot{\theta}\underbrace{\frac{d\hat{\mathbf{e}}_\theta}{dt}}_{-\dot{\theta}\,\hat{\mathbf{e}}_r}$$

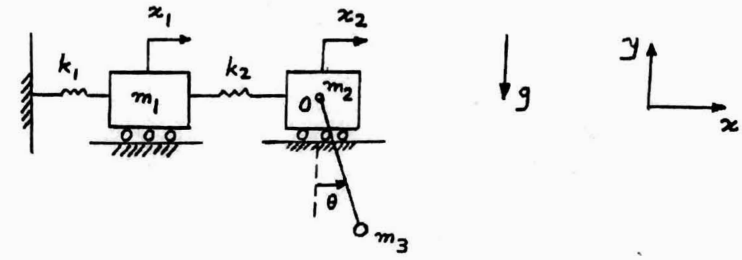
so r-component of $\mathbf{a}_2\big|_r = \ddot{x} - x\dot{\theta}^2$

$$\therefore \qquad m_2 g\cos\theta - k(x - L_0) = m_2\ddot{x} - m_2 x\dot{\theta}^2$$

$$\Longrightarrow \qquad m_2\ddot{x} - m_2 x\dot{\theta}^2 - m_2 g\cos\theta + k(x - L_0) = 0$$

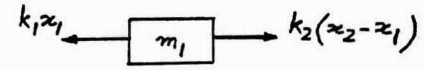
Problem 4

pendulum length = L



Linear momentum in x direction for m_1 :

F.B.D.

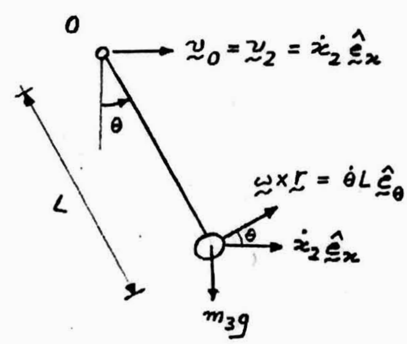


$$m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1x_1$$

$$\Longrightarrow \quad \boxed{m_1\ddot{x}_1 = -(k_1+k_2)x_1 + k_2x_2}$$

Angular momentum about point 0 :

F.B.D.



$$\underset{\sim}{\tau}_0 = \frac{d\underset{\sim}{h}_0}{dt} + \underset{\sim}{v}_0 \times \underset{\sim}{P}$$

$$\underset{\sim}{\tau}_0 = -m_3 g L \sin\theta \ \hat{e}_z$$

$$\underset{\sim}{v}_0 = \underset{\sim}{v}_2 = \dot{x}_2\hat{e}_x$$

$$\underset{\sim}{v}_3 = (\dot{x}_2 + L\dot{\theta}\cos\theta)\hat{e}_x + (L\dot{\theta}\sin\theta)\hat{e}_y$$

$$\underset{\sim}{h}_0 = \underset{\sim}{r}\times\underset{\sim}{P} = \underset{\sim}{r}\times m_3\underset{\sim}{v}_3 = m_3\left(L\sin\theta\,\hat{e}_x - L\cos\theta\,\hat{e}_y\right)\times\left[(\dot{x}_2 + L\dot{\theta}\cos\theta)\hat{e}_x + (L\dot{\theta}\sin\theta)\hat{e}_y\right]$$

$$= m_3\left(L^2\dot{\theta} + \dot{x}_2 L\cos\theta\right)\hat{e}_z$$

$$\underset{\sim}{v}_0\times\underset{\sim}{P} = \underset{\sim}{v}_0\times m_3\underset{\sim}{v}_3 = m_3 L\dot{x}_2\dot{\theta}\sin\theta\ \underset{\sim}{e}_z$$

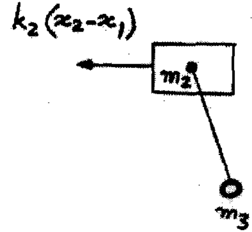
$$\therefore \quad -m_3 gL\sin\theta = m_3\left(L^2\ddot{\theta} + \ddot{x}_2 L\cos\theta - \cancel{\dot{x}_2 L\sin\theta\,\dot{\theta}}\right) + \cancel{m_3 L\dot{x}_2\dot{\theta}\sin\theta}$$

$$\Longrightarrow \quad \boxed{L\ddot{\theta} + \ddot{x}_2\cos\theta + g\sin\theta = 0}$$

Problem 4

Linear momentum in x direction for m_2 & m_3:

F.B.D.



$$\underset{\sim}{a}_2 = \ddot{x}_2 \hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{a}_3\Big|_x = \ddot{x}_2 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta$$

$$-k_2(x_2-x_1) = m_2\ddot{x}_2 + m_3\left(\ddot{x}_2 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right)$$

$$\Longrightarrow \quad (m_2+m_3)\ddot{x}_2 + m_3 L\ddot{\theta}\cos\theta - m_3 L\dot{\theta}^2\sin\theta + k_2(x_2-x_1) = 0$$