2.032 DYNAMICS Fall 2009

Problem Set No. 9

Out: Wednesday, November 4, 2009

The homework problems are for practice only. Solutions are posted in a separate file. Please work on the problems and be prepared to ask questions related to this homework in the recitation of Tuesday, November 10, 2009 (4:00–5:30pm in Room 1-379).

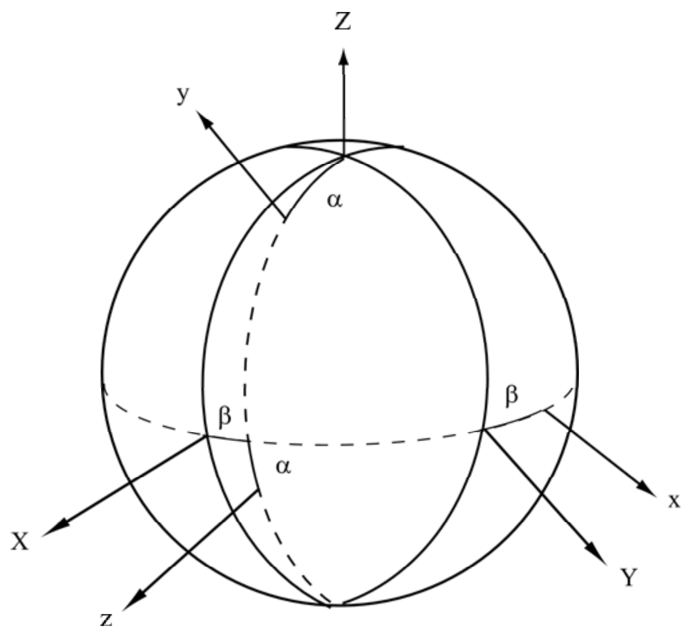
Problem 1

We wish to examine the stability to small perturbations of a spinning top with its symmetry axis along the vertical. This configuration is sometimes referred to as a 'sleeping top' because a smooth symmetric top with its axis vertical might appear to be not moving at all.

Note that, in the undisturbed state, the symmetry (z-) axis coincides with the vertical (Z-) axis and the azimuthal angle ϕ is not well defined. As a result, the usual Euler angles θ, ϕ and ψ, as defined in class, are not convenient coordinates for the purpose of studying the stability of a sleeping top because slight tipping of the axis of the top does not correspond to a small change in ϕ necessarily. One way to handle this difficulty is to let

$$\phi = \frac{\pi}{2} + \beta, \qquad \theta = \frac{\pi}{2} + \alpha$$

with $\alpha, \beta \ll 1$. Thus, the symmetry (z-) axis of the top is close to the X-axis, and now small changes in the orientation of the top axis correspond to $\alpha, \beta \ll 1$ (see the figure below). Of course, in order to properly account for the effect of gravity, the gravitational acceleration has to be taken along the $-X$ direction, $\mathbf{g} = -g\hat{\mathbf{e}}_{\mathbf{X}}$.



(a) Show that the linearized equations of motion for α, β can be written in the following matrix form

$$I\left\{\begin{array}{c}\ddot{\alpha}\\ \ddot{\beta}\end{array}\right\}+H_z\left[\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right]\left\{\begin{array}{c}\dot{\alpha}\\ \dot{\beta}\end{array}\right\}-mgL\left\{\begin{array}{c}\alpha\\ \beta\end{array}\right\}=0,$$

using the same notation as in class.

(b) Look for separable solutions of these equations

$$\left\{\begin{array}{c}\alpha\\ \beta\end{array}\right\}=\left\{\begin{array}{c}a\\ b\end{array}\right\}e^{pt}.$$

Solve for p and show that the spin $n = H_z/I_3$ of the top has to exceed the critical value

$$n_{\text{crit}} = \frac{2}{I_3}(ImgL)^{1/2}$$

for the vertical position to be stable.

(c) Assuming that $n > n_{\text{crit}}$ so that the vertical position is stable, solve for the amplitude ratio a/b corresponding to each (imaginary) value of p.

(d) Based on the results in (c), discuss the nature of small motions about the vertical of a stable sleeping top.