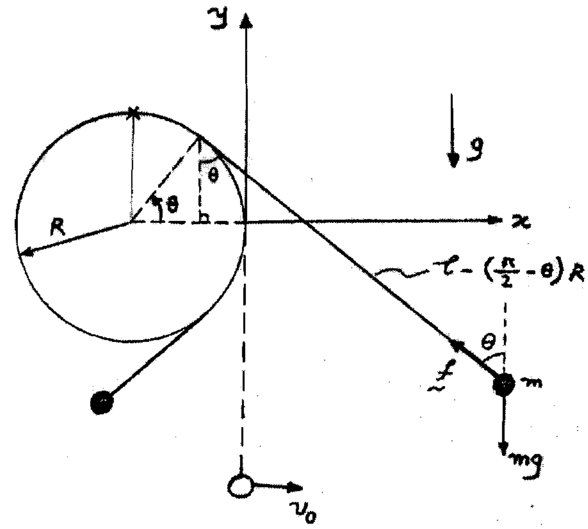
Problem Set No. 1

Problem 1



$$\begin{cases} x_m = \left[\ell - \left(\frac{\pi}{2} - \theta\right) R\right] \sin\theta - R(1 - \cos\theta) \\ y_m = -\left[\ell - \left(\frac{\pi}{2} - \theta\right) R\right] \cos\theta + R \sin\theta \end{cases}$$

$$\begin{cases} \dot{x}_m = \left[\ell - \left(\frac{\pi}{2} - \theta\right) R\right] \cos\theta . \dot{\theta} \\ \dot{y}_m = \left[\ell - \left(\frac{\pi}{2} - \theta\right) R\right] \sin\theta . \dot{\theta} \end{cases}$$

$$\underset{\sim}{v}_m = \dot{x}_m \underset{\sim}{\hat{e}}_x + \dot{y}_m \underset{\sim}{\hat{e}}_y$$

$$\underset{\sim}{f} = |\underset{\sim}{f}| \left(-\sin\theta \, \underset{\sim}{\hat{e}}_x + \cos\theta \, \underset{\sim}{\hat{e}}_y\right)$$

$\therefore \quad \underset{\sim}{f} \cdot \underset{\sim}{v}_m = 0$

$\Longrightarrow$ $\underset{\sim}{f}$ does not work $\Longrightarrow$ The only force that does work is gravitational force which is conservative. $\Longrightarrow$ System is conservative

$\Longrightarrow$ KE + PE = const.

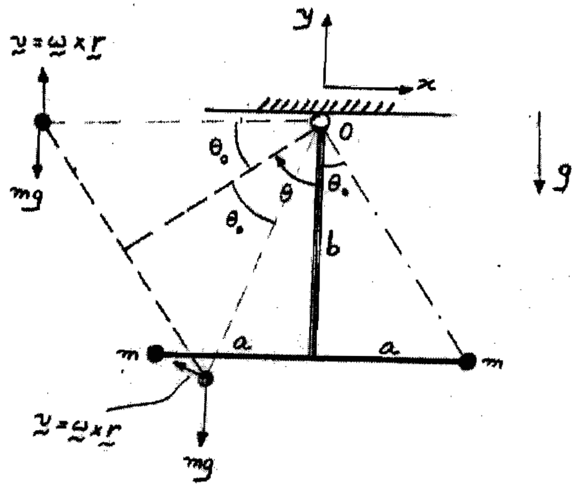
At $\theta = 0$, $\begin{cases} KE = \frac{1}{2} m v_m^2 = \frac{1}{2} m v_0^2 \\ PE = m g y_m(\theta = 0) = -mg\left(\ell - \frac{\pi}{2} R\right) \end{cases}$

At two extreme deflections, $v_m = 0 \Longrightarrow KE = 0$

$$\therefore \quad PE\Big|_{\theta = \theta_{max/min}} = (PE + KE)\Big|_{\theta = 0} = \frac{1}{2} m v_0^2 - mg\left(\ell - \frac{\pi}{2} R\right) = m g y_m\Big|_{\theta = \theta_{max/min}}$$

$$\Longrightarrow \quad y_m\Big|_{\theta = \theta_{max/min}} = \frac{v_0^2}{2g} - \ell + \frac{\pi}{2} R = \left[R \sin\theta - \left[\ell - \left(\frac{\pi}{2} - \theta\right) R\right] \cos\theta\right]_{\theta = \theta_{max}/\theta_{min}}$$

Problem 2



$$\theta_0 = \tan^{-1}\left(\frac{a}{b}\right)$$

$$|\underset{\sim}{r}| = \sqrt{a^2+b^2}$$

$$|\underset{\sim}{\omega}| = \dot{\theta}$$

$$\underset{\sim}{v}_0 = 0 \qquad \Longrightarrow \qquad \underset{\sim}{\tau}_0^{ext} = \frac{d\underset{\sim}{h}_0}{dt}$$

$$\underset{\sim}{\tau}_0 = \left\{ mg\sqrt{a^2+b^2}\ \sin(\theta+\theta_0) + mg\sqrt{a^2+b^2}\ \sin(\theta-\theta_0) \right\} \hat{\underset{\sim}{e}}_z$$

$$= mg\sqrt{a^2+b^2}\ \left(2\sin\theta\cos\theta_0\right) \hat{\underset{\sim}{e}}_z$$

$$|\underset{\sim}{v}| = \sqrt{a^2+b^2}\,\dot{\theta} \qquad \longrightarrow \qquad |\underset{\sim}{P}| = m\sqrt{a^2+b^2}\,\dot{\theta}$$

$$\underset{\sim}{h}_0 = -2\sqrt{a^2+b^2}\left(m\sqrt{a^2+b^2}\,\dot{\theta}\right)\hat{\underset{\sim}{e}}_z$$

Angular momentum about point O :

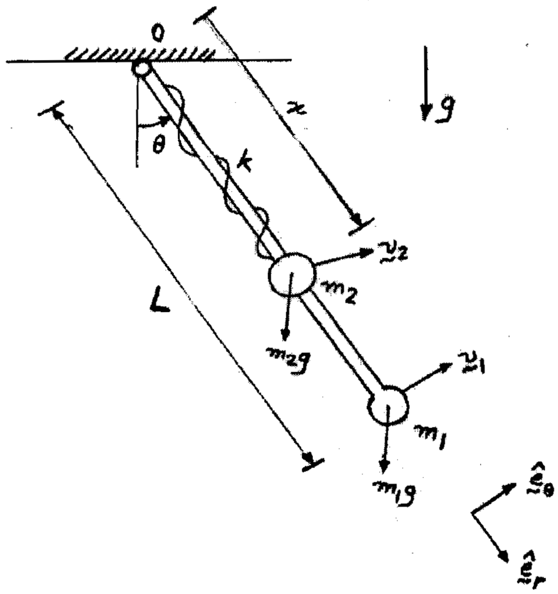
$$2mg\sqrt{a^2+b^2}\ \sin\theta\cos\theta_0 = -2m\left(a^2+b^2\right)\ddot{\theta}$$

$$\cos\theta_0 = \frac{b}{\sqrt{a^2+b^2}}$$

$$\therefore \qquad \underline{\ddot{\theta} + \frac{gb}{a^2+b^2}\sin\theta = 0}$$

equation of motion for $\theta(t)$

Problem 3



Assume free length of spring to be L_0.

a) One needs two coordinates θ and x to describe the motion of the system.

θ and x are a complete and independent set of generalized coordinates.

b) We need to find two equations:

First, angular momentum about point O for the system:

$$\underset{\sim}{\tau}_0^{ext} = \frac{d\underset{\sim}{h}_0}{dt}$$

$$\underset{\sim}{\tau}_0 = -\left(m_1 g L \sin\theta + m_2 g x \sin\theta\right)\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{v}_1 = L\dot{\theta}\,\hat{\underset{\sim}{e}}_\theta \qquad \& \qquad \underset{\sim}{v}_2 = \dot{x}\,\hat{\underset{\sim}{e}}_r + x\dot{\theta}\,\hat{\underset{\sim}{e}}_\theta$$

$$\underset{\sim}{h}_0 = \sum_{i=1}^{2} \underset{\sim}{r}_i \times \underset{\sim}{P}_i = (L\,\hat{\underset{\sim}{e}}_r)\times(m_1 L\dot{\theta}\,\hat{\underset{\sim}{e}}_\theta) + (x\,\hat{\underset{\sim}{e}}_r)\times\left[m_2(\dot{x}\,\hat{\underset{\sim}{e}}_r + x\dot{\theta}\,\hat{\underset{\sim}{e}}_\theta)\right]$$

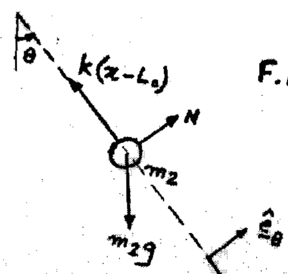
$$\Longrightarrow \underset{\sim}{h}_0 = \left(m_1 L^2\dot{\theta} + m_2 x^2\dot{\theta}\right)\hat{\underset{\sim}{e}}_z$$

$$\therefore \quad -\left(m_1 g L \sin\theta + m_2 g x \sin\theta\right) = \left(m_1 L^2 + m_2 x^2\right)\ddot{\theta} + 2m_2 x\dot{x}\dot{\theta}$$

$$\Longrightarrow \quad \left(m_1 L^2 + m_2 x^2\right)\ddot{\theta} + 2m_2 x\dot{x}\dot{\theta} + \left(m_1 L + m_2 x\right) g\sin\theta = 0$$

To find the second equation, apply linear momentum in the radial direction for m_2:

m_2 is free to slide along the rod so the force N has no component along the radial direction.



F.B.D.

Problem 3

$$\underset{\sim}{v}_2 = \dot{x}\,\hat{\underset{\sim}{e}}_r + x\dot{\theta}\,\hat{\underset{\sim}{e}}_\theta$$

$$\underset{\sim}{a}_2 = \frac{d\underset{\sim}{v}_2}{dt} = \ddot{x}\,\hat{\underset{\sim}{e}}_r + \dot{x}\underbrace{\frac{d\hat{\underset{\sim}{e}}_r}{dt}}_{\dot{\theta}\,\hat{\underset{\sim}{e}}_\theta} + (x\ddot{\theta} + \dot{x}\dot{\theta})\,\hat{\underset{\sim}{e}}_\theta + x\dot{\theta}\underbrace{\frac{d\hat{\underset{\sim}{e}}_\theta}{dt}}_{-\dot{\theta}\,\hat{\underset{\sim}{e}}_r}$$

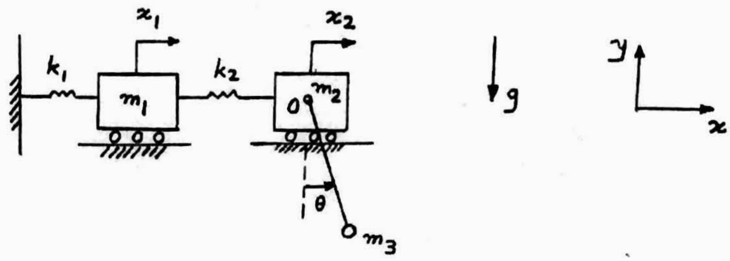
so r-component of $\underset{\sim}{a}_2\big|_r = \ddot{x} - x\dot{\theta}^2$

$$\therefore \qquad m_2 g\cos\theta - k(x - L_0) = m_2\ddot{x} - m_2 x\dot{\theta}^2$$

$$\Longrightarrow \qquad \boxed{m_2\ddot{x} - m_2 x\dot{\theta}^2 - m_2 g\cos\theta + k(x - L_0) = 0}$$

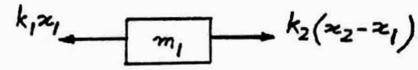
Problem 4

pendulum length = L



Linear momentum in x direction for m_1:

F.B.D.

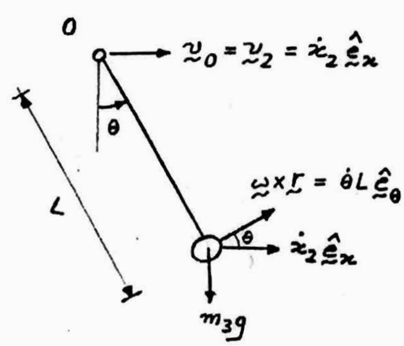


$$m_1 \ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1$$

$$\Longrightarrow \quad \underline{m_1 \ddot{x}_1 = -(k_1 + k_2)x_1 + k_2 x_2}$$

Angular momentum about point 0:

F.B.D.



$$\underset{\sim}{\tau}_0 = \frac{d\underset{\sim}{h}_0}{dt} + \underset{\sim}{v}_0 \times \underset{\sim}{P}$$

$$\underset{\sim}{\tau}_0 = -m_3 g L \sin\theta \ \hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{v}_0 = \underset{\sim}{v}_2 = \dot{x}_2 \hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{v}_3 = (\dot{x}_2 + L\dot{\theta}\cos\theta)\hat{\underset{\sim}{e}}_x + (L\dot{\theta}\sin\theta)\hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{h}_0 = \underset{\sim}{r} \times \underset{\sim}{P} = \underset{\sim}{r} \times m_3 \underset{\sim}{v}_3 = m_3 \left(L\sin\theta\, \hat{\underset{\sim}{e}}_x - L\cos\theta\, \hat{\underset{\sim}{e}}_y\right) \times \left[(\dot{x}_2 + L\dot{\theta}\cos\theta)\hat{\underset{\sim}{e}}_x + (L\dot{\theta}\sin\theta)\hat{\underset{\sim}{e}}_y\right]$$

$$= m_3 \left(L^2\dot{\theta} + \dot{x}_2 L\cos\theta\right)\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{v}_0 \times \underset{\sim}{P} = \underset{\sim}{v}_0 \times m_3 \underset{\sim}{v}_3 = m_3 L \dot{x}_2 \dot{\theta} \sin\theta \ \underset{\sim}{e}_z$$

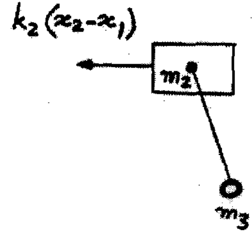
$$\therefore \quad -m_3 g L \sin\theta = m_3\left(L^2\ddot{\theta} + \ddot{x}_2 L\cos\theta - \cancel{\dot{x}_2 L\sin\theta\,\dot{\theta}}\right) + \cancel{m_3 L\dot{x}_2\dot{\theta}\sin\theta}$$

$$\Longrightarrow \quad \underline{L\ddot{\theta} + \ddot{x}_2\cos\theta + g\sin\theta = 0}$$

Problem 4

Linear momentum in x direction for m_2 & m_3:

F.B.D.



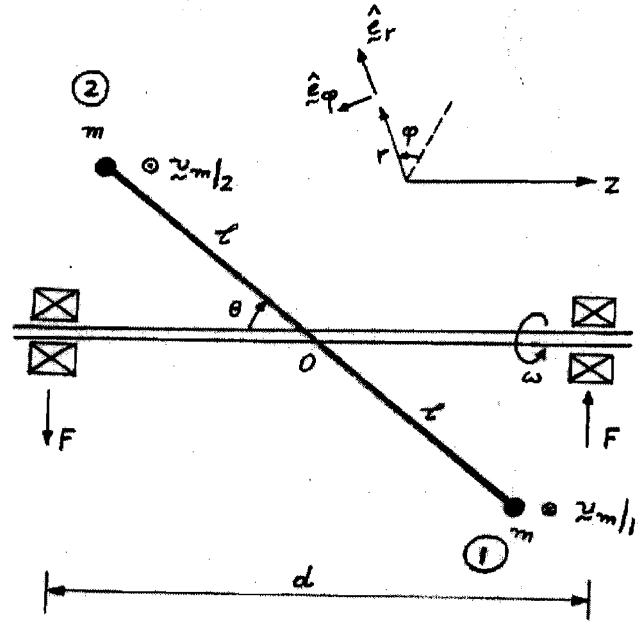
$$\underset{\sim}{a}_2 = \ddot{x}_2 \hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{a}_3\Big|_x = \ddot{x}_2 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta$$

$$-k_2(x_2-x_1) = m_2\ddot{x}_2 + m_3\left(\ddot{x}_2 + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta\right)$$

$$\Longrightarrow \quad (m_2+m_3)\ddot{x}_2 + m_3L\ddot{\theta}\cos\theta - m_3L\dot{\theta}^2\sin\theta + k_2(x_2-x_1) = 0$$

Problem 5



$$v_m|_1 = v_m|_2 = \omega \times r$$

$$= \omega \ell \sin\theta\, \hat{e}_\varphi$$

Apply angular momentum about point O:

$$h_0 = r \times P = (\ell\cos\theta\,\hat{e}_z + \ell\sin\theta\,\hat{e}_r) \times m\omega\ell\sin\theta\,\hat{e}_\varphi$$

$$+ (-\ell\cos\theta\,\hat{e}_z + \ell\sin\theta\,\hat{e}_r) \times m\omega\ell\sin\theta\,\hat{e}_\varphi$$

$$= -m\omega\ell^2\sin\theta\cdot\cos\theta\,\hat{e}_r + m\omega\ell^2\sin^2\theta\,\hat{e}_z$$

$$+ m\omega\ell^2\sin\theta\cos\theta\,\hat{e}_r + m\omega\ell^2\sin^2\theta\,\hat{e}_z$$

$$\frac{dh_0}{dt} = -m\omega^2\ell^2\sin\theta\cos\theta\,\hat{e}_\varphi|_1 + m\omega^2\ell^2\sin\theta\cos\theta\,\hat{e}_\varphi|_2$$

(Note that $\frac{d\hat{e}_r}{dt} = \omega\,\hat{e}_\varphi$; $\hat{e}_\varphi|_1$ and $\hat{e}_\varphi|_2$ are in opposite directions)

$$\tau_0^{ext} = Fd$$

$$\tau_0^{ext} = \frac{dh_0}{dt} \implies Fd = 2m\omega^2\ell^2\sin\theta\cos\theta = m\omega^2\ell^2\sin 2\theta$$

$$\implies F = \frac{m\omega^2\ell^2}{d}\sin 2\theta \implies F_{max} = \frac{m\omega^2\ell^2}{d} \text{ for the case of } \theta = 45^\circ$$

Note that force F changes direction but is always in the plane of the shaft and the rods.