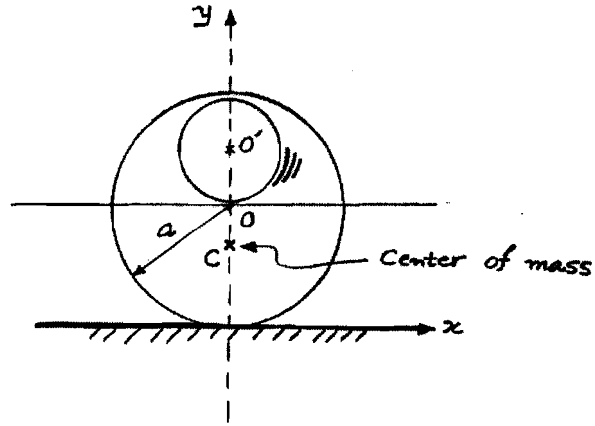
2.032 Dynam

Problem Set No. 3

Problem 1



Assume

Length of the cylinder $= h$

Density $= \rho$

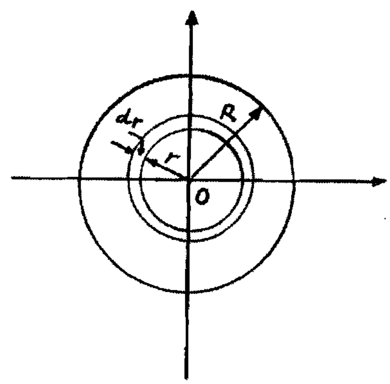
So, $M = \rho h \left[\pi a^2 - \pi\left(\frac{a}{2}\right)^2\right] = \frac{3}{4}\pi a^2 \rho h$

(i)

$x_c = 0$ because of symmetry

$$y_c = \frac{\sum m_i y_i}{\sum m_i} = \frac{\pi a^2 (a) - \pi\left(\frac{a}{2}\right)^2\left(a + \frac{a}{2}\right)}{\pi a^2 - \pi\left(\frac{a}{2}\right)^2} = \frac{5}{6}a$$

$$\Rightarrow \quad co = \frac{a}{6} \quad \& \quad co' = \frac{a}{6} + \frac{a}{2} = \frac{2}{3}a$$



$$I_0 = \sum_i m_i r_i^2 = \rho h \int_0^R 2\pi r\, dr\, (r^2) = \frac{\pi}{2} R^4 \rho h$$

Finding moment of inertia using Parallel-axes theorem:

$$I_c = \rho h \left\{ \frac{\pi}{2} a^4 + (\pi a^2)\, \overset{\left(\frac{a}{6}\right)^2}{co^2} - \left[\frac{\pi}{2}\left(\frac{a}{2}\right)^4 + \pi\left(\frac{a}{2}\right)^2 \overset{\left(\frac{2a}{3}\right)^2}{co'^2} \right] \right\} = \frac{37}{96}\pi a^4 \rho h$$

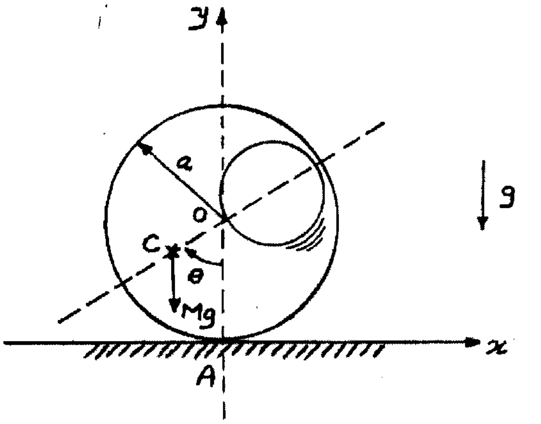
Problem 1

(i)

$$PE = Mg\,y_C = \frac{3}{4}\pi a^2 \rho g h \left(a - \overset{\frac{a}{6}}{OC}\cdot \cos\theta\right)$$

$$\boxed{PE = \frac{3}{4}\rho g h \pi a^3 \left(1 - \frac{\cos\theta}{6}\right)}$$

Potential energy of the cylinder



$$KE = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2$$

$$\underset{\sim}{\omega} = -\dot\theta\, \underset{\sim}{\hat e_z}$$

$$\underset{\sim}{v_C} = \cancelto{0}{\underset{\sim}{v_A}} + \underset{\sim}{\omega} \times \underset{\sim}{AC} \longrightarrow v_C = \dot\theta (AC) = \dot\theta \sqrt{OC^2 + OA^2 - 2(OC)(OA)\cos\theta}$$

$$\longrightarrow v_C = \dot\theta \sqrt{\left(\frac{a}{6}\right)^2 + a^2 - 2\left(\frac{a}{6}\right)(a)\cos\theta} = a\dot\theta \sqrt{\frac{37}{36} - \frac{\cos\theta}{3}}$$

$$\therefore \quad KE = \frac{1}{2}\left(\frac{3}{4}\pi a^2 \rho h\right)\left[a^2\dot\theta^2\left(\frac{37}{36} - \frac{\cos\theta}{3}\right)\right] + \frac{1}{2}\left(\frac{37}{96}\pi a^4 \rho h\right)\dot\theta^2$$

$$\boxed{KE = \pi \rho h a^4 \dot\theta^2 \left(\frac{37}{64} - \frac{\cos\theta}{8}\right)}$$

Kinetic energy of the cylinder

(ii) conservative system $\longrightarrow \frac{d}{dt}(KE + PE) = 0$

Assume θ and $\dot\theta$ are small $\left(\sin\theta \approx \theta, \ \cos\theta \approx 1 - \frac{\theta^2}{2}\right)$:

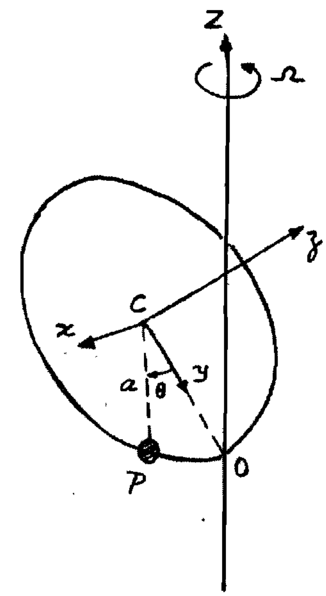
Keeping up to quadratic term in $\theta, \dot\theta$,

$$KE \approx \pi \rho h a^4 \dot\theta^2 \left(\frac{29}{64}\right) \quad \& \quad PE \approx \frac{3}{4}\rho g h \pi a^3 \left(\frac{5}{6} + \frac{\theta^2}{12}\right)$$

$$\frac{d}{dt}(KE + PE) = 0 \implies \frac{29}{32} a\ddot\theta + \frac{1}{8} g\theta = 0 \longrightarrow \omega_n^2 = \frac{4}{29}\frac{g}{a}$$

$$\therefore \quad \text{natural frequency} \quad \omega_n = 2\sqrt{\frac{g}{29a}}$$

Problem 2



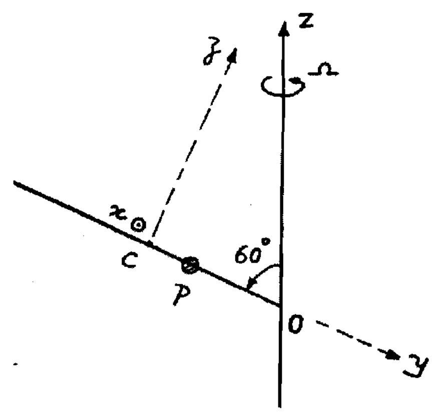
Introducing xyz coordinate system fixed to the ring and as a result rotating with Ω about Z axis:

$$\mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_P\big|_{\text{rel. to } C}$$

$$\mathbf{v}_C = \frac{d}{dt}\mathbf{OC} = -\Omega\,\mathbf{e}_Z \times \mathbf{OC}$$

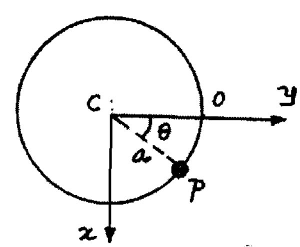
$$= \Omega\left(\frac{\sqrt{3}}{2}\hat{\mathbf{e}}_z - \frac{1}{2}\hat{\mathbf{e}}_y\right) \times \left(-a\,\hat{\mathbf{e}}_y\right)$$

$$\rightarrow \quad \mathbf{v}_C = a\Omega\frac{\sqrt{3}}{2}\,\hat{\mathbf{e}}_x$$



$$\mathbf{v}_P\big|_{\text{rel. to } C} = \frac{d}{dt}\mathbf{CP} = \boldsymbol{\omega}\big|_{\mathbf{CP}} \times \mathbf{CP}$$

$$\boldsymbol{\omega}\big|_{\mathbf{CP}} = \Omega\,\hat{\mathbf{e}}_Z - \dot{\theta}\,\hat{\mathbf{e}}_z = -\frac{\Omega}{2}\hat{\mathbf{e}}_y + \left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)\hat{\mathbf{e}}_z$$



$$\mathbf{CP} = a\sin\theta\,\hat{\mathbf{e}}_x + a\cos\theta\,\hat{\mathbf{e}}_y$$

$$\Longrightarrow \quad \mathbf{v}_P\big|_{\text{rel. to } C} = \left[-\frac{\Omega}{2}\hat{\mathbf{e}}_y + \left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)\hat{\mathbf{e}}_z\right] \times \left[a\sin\theta\,\hat{\mathbf{e}}_x + a\cos\theta\,\hat{\mathbf{e}}_y\right]$$

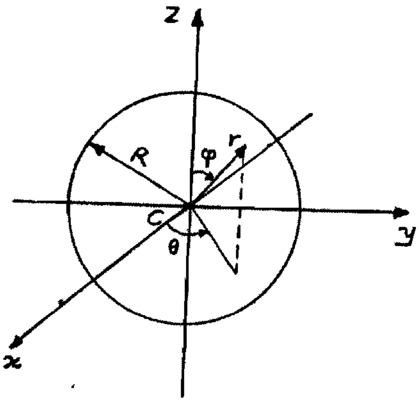
$$= \frac{\Omega}{2}a\sin\theta\,\hat{\mathbf{e}}_z + a\sin\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)\hat{\mathbf{e}}_y - a\cos\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)\hat{\mathbf{e}}_x$$

$$\therefore \quad \mathbf{v}_P = \left[a\Omega\frac{\sqrt{3}}{2}(1-\cos\theta) + a\dot{\theta}\cos\theta\right]\hat{\mathbf{e}}_x + \left[a\sin\theta\left(\Omega\frac{\sqrt{3}}{2} - \dot{\theta}\right)\right]\hat{\mathbf{e}}_y + \left[a\frac{\Omega}{2}\sin\theta\right]\hat{\mathbf{e}}_z$$

Velocity of P in terms of θ and $\dot{\theta}$

Problem 3

(a) a sphere of radius R,



It is convenient to use spherical coordinate system :

$$x = r \sin\varphi \cos\theta$$

$$y = r \sin\varphi \sin\theta$$

$$z = r \cos\varphi$$

$$dV = r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

$$M = \frac{4}{3}\pi\rho R^3$$

Because of symmetry : $I_x = I_y = I_z$ & $\underline{I_{xy} = I_{yz} = I_{zx} = 0}$

$$I_z = \int \rho \, dV (x^2+y^2) = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{R} dr \; \rho r^2 \sin\varphi \; (r^2 \sin^2\varphi)$$

$$= 2\pi\rho \int_0^{\pi} d\varphi \int_0^{R} dr \; r^4 \sin^3\varphi = 2\pi\rho \frac{R^5}{5} \int_0^{\pi} \sin^3\varphi \, d\varphi \qquad \sin^3\varphi = \frac{1}{4}(3\sin\varphi - \sin 3\varphi)$$

$$= 2\pi\rho \frac{R^5}{5}\left[-\frac{3\cos\varphi}{4} + \frac{\cos 3\varphi}{12}\right]_0^{\pi} = \frac{8\pi}{15}\rho R^5 = \underline{\frac{2}{5} M R^2}$$

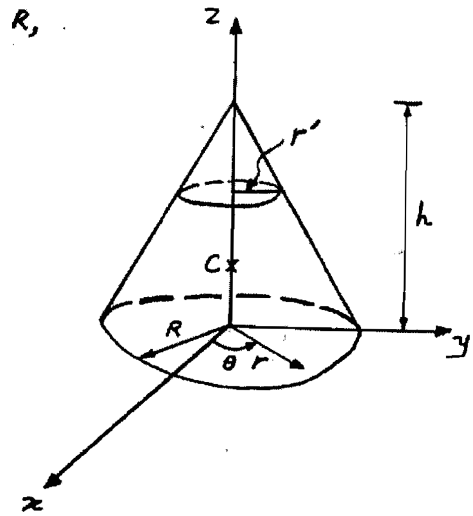
$$\underline{I_x = I_y = \frac{2}{5} M R^2}$$

$$\therefore \quad [I]_C = MR^2 \begin{bmatrix} \frac{2}{5} & 0 & 0 \\ & \frac{2}{5} & 0 \\ & & \frac{2}{5} \end{bmatrix} \qquad \text{centroidal moments of inertia}$$

clearly, $Cxyz$ are principal axes at C and I_x, I_y & I_z are principal centroidal moments of inertia. Due to symmetry, any axis passing through C is a principal axis with principal moment $\frac{2}{5}MR^2$.

Problem 3

(b) a circular cone of height h and base radius R,



It is convenient to use cylindrical coordinate system:

$$x = r\cos\theta$$

$$y = r\sin\theta \qquad\qquad dV = r\,dr\,d\theta\,dz$$

$$z = z$$

$$r' = \left(1-\frac{z}{h}\right)R$$

$$M = \frac{1}{3}\rho\pi R^2 h$$

Because of symmetry: $\begin{cases} x_c = y_c = 0 \\ I_{xy} = I_{yz} = I_{zx} = 0 \end{cases}$ & $I_x = I_y$

$$z_c = \frac{\int_0^h \rho\pi r'^2 dz\, z}{\frac{1}{3}\rho\pi R^2 h} = \frac{\rho\pi\int_0^h \left(1-\frac{z}{h}\right)^2 R^2 z\,dz}{\frac{1}{3}\rho\pi R^2 h} = \frac{h}{4}$$

$$I_z = \int \rho\,dV\,(x^2+y^2) = \rho\int_0^{2\pi} d\theta \int_0^h dz \int_0^{r'} r\,dr\,(r^2)$$

$$= 2\pi\rho\int_0^h dz \int_0^{(1-\frac{z}{h})R} r^3 dr = 2\pi\rho\frac{R^4}{4}\int_0^h dz\left(1-\frac{z}{h}\right)^4 = \frac{\pi}{10}\rho R^4 h = \frac{3}{10}MR^2$$

$$I_x = \int\rho\,dV\,(y^2+z^2) = \rho\int_0^{2\pi} d\theta\int_0^h dz\int_0^{r'} r\,dr\,(r^2\sin^2\theta + z^2)$$

$$= \rho\int_0^{2\pi} d\theta\int_0^h dz\int_0^{(1-\frac{z}{h})R}(r^3\sin^2\theta + rz^2)\,dr = \rho\int_0^{2\pi} d\theta\int_0^h dz\left[\frac{(1-\frac{z}{h})^4 R^4}{4}\sin^2\theta + \frac{(1-\frac{z}{h})^2 R^2 z^2}{2}\right]$$

$$= \rho\int_0^{2\pi} d\theta\left(\frac{R^4 h\sin^2\theta}{20} + \frac{h^3R^2}{60}\right) = \rho\frac{\pi R^4 h}{20} + \rho\frac{\pi h^3 R^2}{30} = \frac{3}{20}MR^2 + M\frac{h^2}{10}$$

$$I_y = \frac{3}{20}MR^2 + \frac{Mh^2}{10}$$

Problem 3

Using parallel-axes theorem $\left(a = b = 0 \,, \quad c = z_C = \frac{h}{4} \right)$:

$$[I]_C = \begin{bmatrix} \frac{3}{20}MR^2 + \frac{1}{10}Mh^2 & 0 & 0 \\ & \frac{3}{20}MR^2 + \frac{1}{10}Mh^2 & 0 \\ & & \frac{3}{10}MR^2 \end{bmatrix} - M \begin{bmatrix} \left(\frac{h}{4}\right)^2 & 0 & 0 \\ & \left(\frac{h}{4}\right)^2 & 0 \\ & & 0 \end{bmatrix}$$

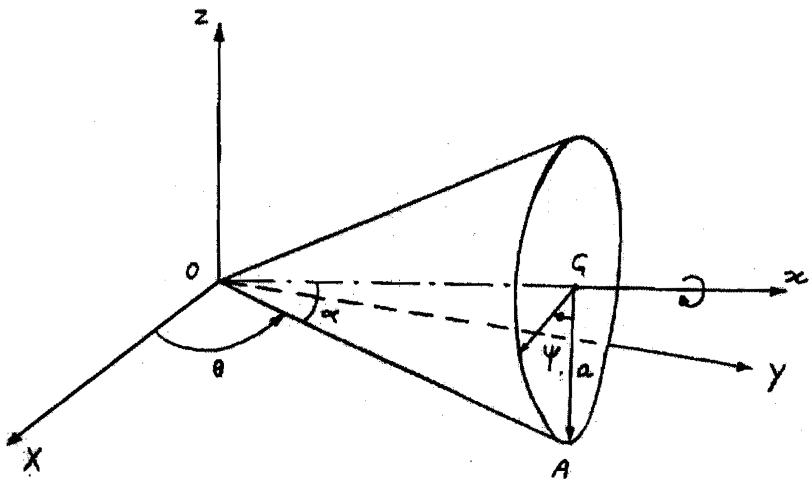
$$[I]_C = \begin{bmatrix} \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right) & 0 & 0 \\ & \frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right) & 0 \\ & & \frac{3}{10}MR^2 \end{bmatrix}$$

principal centroidal moments of inertia

Because of symmetry, any axis passing through C and parallel to xy plane is a principal axis with principal moment $\frac{3}{20}M\left(R^2 + \frac{1}{4}h^2\right)$.

Problem 4

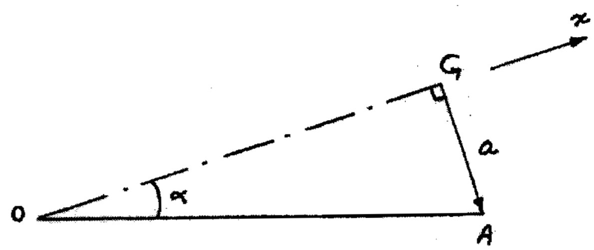
$\underset{\sim}{\omega}_{cone} = \dot\theta \, \hat{\underset{\sim}{e}}_Z - \dot\psi \, \hat{\underset{\sim}{e}}_x$



Impose no slip : $\underset{\sim}{v}_A \big|_{cone} = \underset{\sim}{0}$

$\underset{\sim}{v}_A \big|_{cone} = \underset{\sim}{v}_O \big|_{cone} + \underset{\sim}{\omega}\big|_{cone} \times \underset{\sim}{OA} = \underset{\sim}{0}$

$\underset{\sim}{v}_O \big|_{cone} = \underset{\sim}{0}$



$\underset{\sim}{OA} = \dfrac{a}{\sin\alpha} \left(\cos\theta \, \hat{\underset{\sim}{e}}_X + \sin\theta \, \hat{\underset{\sim}{e}}_Y \right)$

$\underset{\sim}{e}_x = \cos\alpha \cos\theta \, \hat{\underset{\sim}{e}}_X + \cos\alpha \sin\theta \, \hat{\underset{\sim}{e}}_Y + \sin\alpha \, \hat{\underset{\sim}{e}}_Z$

$$\therefore \quad \underset{\sim}{\omega}\big|_{cone} \times \underset{\sim}{OA} = \underset{\sim}{0} = \left[-\dot\psi \cos\alpha \cos\theta \, \hat{\underset{\sim}{e}}_X - \dot\psi \cos\alpha \sin\theta \, \hat{\underset{\sim}{e}}_Y + \left(\dot\theta - \dot\psi \sin\alpha \right) \hat{\underset{\sim}{e}}_Z \right] \times$$

$$\left[\frac{a}{\sin\alpha} \cos\theta \, \hat{\underset{\sim}{e}}_X + \frac{a}{\sin\alpha} \sin\theta \, \hat{\underset{\sim}{e}}_Y \right]$$

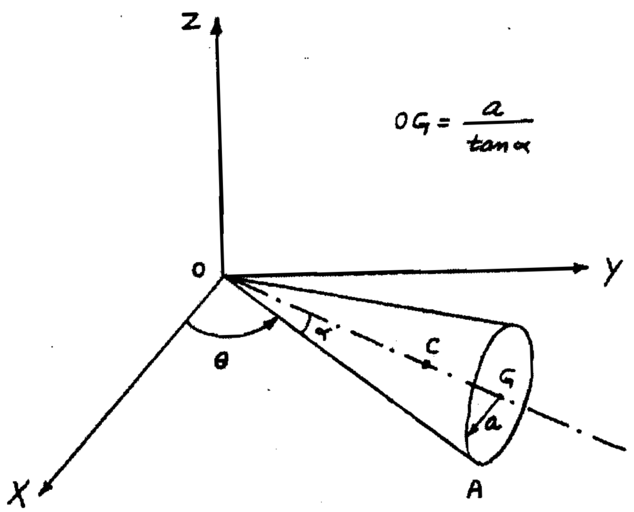
$$= -\hat{\underset{\sim}{e}}_X \left(a\dot\theta \frac{\sin\theta}{\sin\alpha} - a\dot\psi \sin\theta \right) + \hat{\underset{\sim}{e}}_Y \left(a\dot\theta \frac{\cos\theta}{\sin\alpha} - \dot\psi a \cos\theta \right) = \underset{\sim}{0}$$

$$\Longrightarrow \quad \dot\psi = \frac{\dot\theta}{\sin\alpha}$$

Finally, $\underset{\sim}{\omega}_{cone} = \dot\theta \, \hat{\underset{\sim}{e}}_Z - \dfrac{\dot\theta}{\sin\alpha} \hat{\underset{\sim}{e}}_x = -\dot\theta \operatorname{Cotg}\alpha \left(\cos\theta \, \hat{\underset{\sim}{e}}_X + \sin\theta \, \hat{\underset{\sim}{e}}_Y \right) = -\dot\theta \operatorname{Cotg}\alpha \, \hat{\underset{\sim}{e}}_{OA}$

ω_{cone} and axis of rotation are along OA.

...blem 5



...rincipal centroidal moments :

$$I_x = I_y = \frac{3}{20} M \left(a^2 + \frac{a^2}{4\tan^2\alpha} \right)$$

$$I_z = \frac{3}{10} M a^2$$

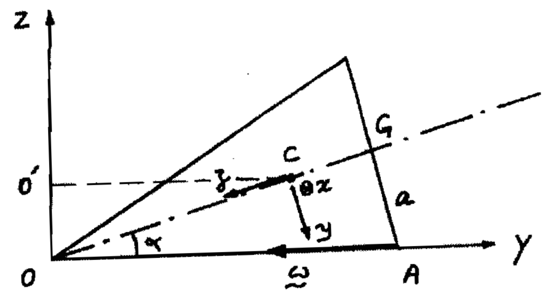
$$T = \frac{1}{2} M \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c + \frac{1}{2} \{\omega\}^t [I]_c \{\omega\}$$

$$\underset{\sim}{v}_c = (\dot\theta \,\hat{\underset{\sim}{e}}_z) \times \underset{\sim}{OC} = \dot\theta\,(O'C)\,\hat{\underset{\sim}{e}}_x$$

$$= \dot\theta\,(OC)\cos\alpha\,\hat{\underset{\sim}{e}}_x$$

$$= \dot\theta \left(\frac{3}{4} OG\right) \cos\alpha\,\hat{\underset{\sim}{e}}_x$$

$$= \frac{3}{4} a \dot\theta \frac{\cos^2\alpha}{\sin\alpha}\,\hat{\underset{\sim}{e}}_x$$



$$\underset{\sim}{v}_c \cdot \underset{\sim}{v}_c = \frac{9}{16} a^2 \dot\theta^2 \frac{\cos^4\alpha}{\sin^2\alpha}$$

$$\underset{\sim}{\omega}_{cone} = -\frac{\dot\theta}{\tan\alpha}\,\hat{\underset{\sim}{e}}_{OA} = -\frac{\dot\theta}{\tan\alpha}\left(-\cos\alpha\,\hat{\underset{\sim}{e}}_z + \sin\alpha\,\hat{\underset{\sim}{e}}_y\right) = -\dot\theta\cos\alpha\,\hat{\underset{\sim}{e}}_y + \dot\theta\frac{\cos^2\alpha}{\sin\alpha}\,\hat{\underset{\sim}{e}}_z$$

$$\Longrightarrow \quad \omega_x = 0 \;, \quad \omega_y = -\dot\theta\cos\alpha \;, \quad \omega_z = \dot\theta\frac{\cos^2\alpha}{\sin\alpha}$$

$$\frac{1}{2}\{\omega\}^t [I]_c \{\omega\} = \frac{1}{2}\left\{ I_y \omega_y^2 + I_z \omega_z^2 \right\}$$

$$\therefore \quad T = \frac{1}{2} M \left(\frac{9}{16} a^2 \dot\theta^2 \frac{\cos^4\alpha}{\sin^2\alpha} \right) + \frac{1}{2} \left\{ \frac{3}{20} M a^2 \left(1 + \frac{1}{4\tan^2\alpha}\right) \dot\theta^2 \cos^2\alpha + \frac{3}{10} M a^2 \dot\theta^2 \frac{\cos^4\alpha}{\sin^2\alpha} \right\}$$

$$\Longrightarrow \quad T = M a^2 \dot\theta^2 \cot^2\alpha \, \frac{3 + 15\cos^2\alpha}{40}$$

kinetic energy of the rolling cone