2.032 Dynamics

Problem Set No. 4

Problem 1

$$[I]_G = \begin{bmatrix} 9 & 4 & 0 \\ & 3 & 0 \\ & & 5 \end{bmatrix} \qquad G: \text{ origin of the coordinate system}$$

Principal moments of inertia are the eigenvalues of $[I]_G$:

To find the eigenvalues I of $[I]_G$,

$$\det \begin{bmatrix} 9-I & 4 & 0 \\ & 3-I & 0 \\ & & 5-I \end{bmatrix} = 0 \qquad \Longrightarrow \qquad (9-I)(3-I)(5-I) - 16(5-I) = 0$$

$$\Longrightarrow \qquad \underline{I_1 = 11}\ , \quad \underline{I_2 = 5}\ , \quad \underline{I_3 = 1} \qquad \text{principal moments of inertia}$$

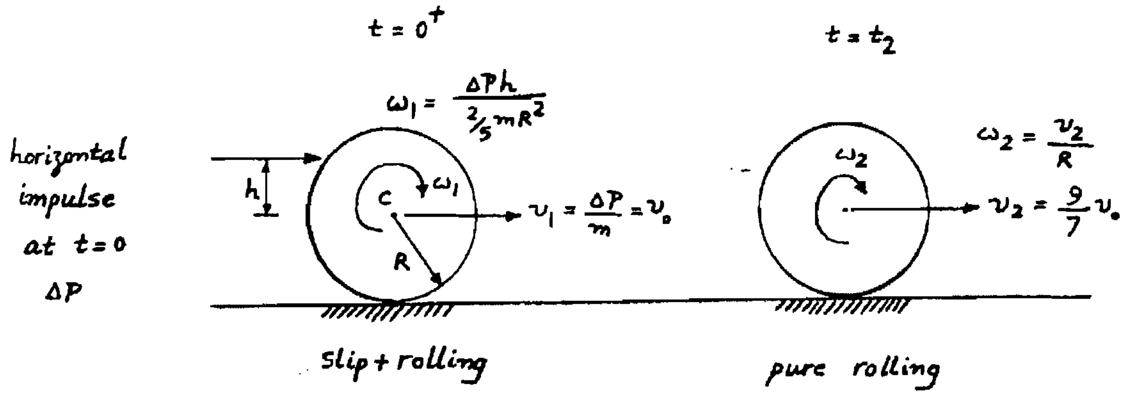
To find direction of principal axis,

for $I_1 = 11$,
$$\begin{bmatrix} 9-11 & 4 & 0 \\ & 3-11 & 0 \\ & & 5-11 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}_1 = \{0\} \quad \Longrightarrow \quad \begin{matrix} \omega_x = 2\omega_y \\ \omega_z = 0 \end{matrix} \quad \Longrightarrow \quad \underset{\sim}{\omega}_1 = \underline{\begin{Bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{Bmatrix}}$$

for $I_2 = 5$,
$$\begin{bmatrix} 9-5 & 4 & 0 \\ & 3-5 & 0 \\ & & 5-5 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}_2 = \{0\} \quad \Longrightarrow \quad \begin{matrix} \omega_x = \omega_y = 0 \\ \omega_z \neq 0 \end{matrix} \quad \Longrightarrow \quad \underset{\sim}{\omega}_2 = \underline{\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}}$$

for $I_3 = 1$,
$$\begin{bmatrix} 9-1 & 4 & 0 \\ & 3-1 & 0 \\ & & 5-1 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}_3 = \{0\} \quad \Longrightarrow \quad \begin{matrix} \omega_y = -2\omega_x \\ \omega_z = 0 \end{matrix} \quad \Longrightarrow \quad \underset{\sim}{\omega}_3 = \underline{\begin{Bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{Bmatrix}}$$

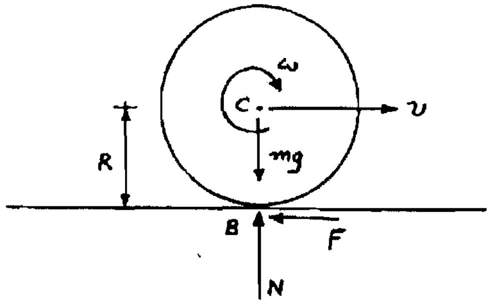
Problem 2



horizontal impulse at $t=0$ ΔP ; ΔP = change in linear momentum $= m v_1 = m v_0$

$\Delta P . h$ = change in angular momentum about $C = I\omega_1 = \frac{2}{5} m R^2 \omega_1$

Apply angular momentum principle about (moving) contact point B :



$$\underset{\sim}{\tau}_B = \frac{d}{dt} \underset{\sim}{H}_B + \underset{\sim}{v}_B \times \underset{\sim}{P}$$

$$\left.\begin{array}{l} \underset{\sim}{P} = m \underset{\sim}{v} \\ \underset{\sim}{v}_B \parallel \underset{\sim}{v} \end{array}\right\} \Longrightarrow \underset{\sim}{v}_B \times \underset{\sim}{P} = \underset{\sim}{0}$$

$$\underset{\sim}{\tau}_B = 0$$

$\therefore$ $\frac{d}{dt} \underset{\sim}{H}_B = 0 \longrightarrow \underset{\sim}{H}_B$ is conserved. $\longrightarrow \left. \underset{\sim}{H}_B \right|_{t=0^+} = \left. H_B \right|_{t=t_2}$

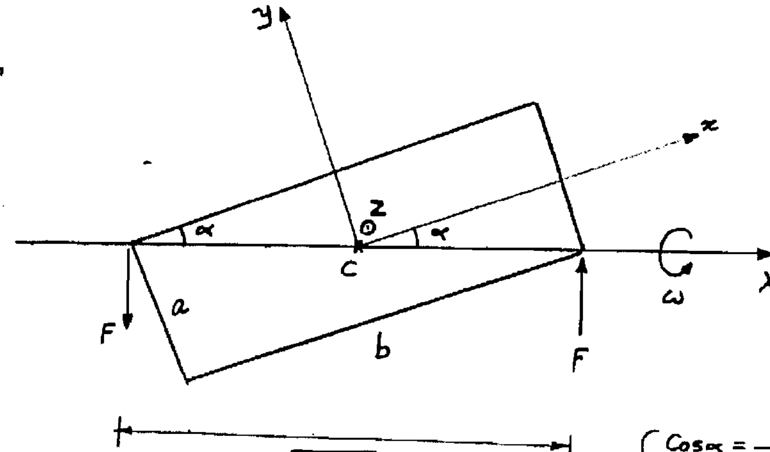
$$\underset{\sim}{H}_B = \underset{\sim}{H}_C + \underset{\sim}{b}_C \times \underset{\sim}{P} = \frac{2}{5} m R^2 \omega + m R v$$

$$\left.\begin{array}{ll} t = 0^+ , & \left. H_B \right|_{t=0^+} = \frac{2}{5} m R^2 \omega_1 + m R v_1 = \Delta P . h + m R v_0 = m v_0 (h + R) \\ t = t_2 , & \left. H_B \right|_{t=t_2} = \frac{2}{5} m R^2 \omega_2 + m R v_2 = \left(\frac{2}{5} m R + m R \right) \frac{9}{7} v_0 = \frac{9}{5} v_0 m R \end{array}\right\} \Longrightarrow h + R = \frac{9}{5} R$$

$$\therefore \quad h = \frac{4}{5} R$$

Problem 3

Introducing xyz coordinate system which is fixed to the plate and rotates with ω about X axis:



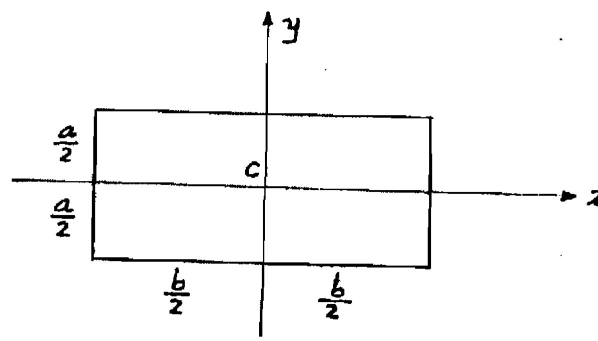
$$\begin{cases} \cos\alpha = \dfrac{b}{\sqrt{a^2+b^2}} \\ \sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}} \end{cases}$$

(a)

Since $\underset{\sim}{r}_c = 0$, forces on the bearings are equal and in opposite directions.

$$I_x = \int(\rho dv)(y^2+z^2) \qquad z^2 \approx 0 \quad \text{(thin plate)}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}} \rho\, dx\, dy\, (y^2) = \rho b \frac{a^3}{12} = \underline{\frac{1}{12} M a^2}$$



$$\underline{I_y = \frac{1}{12} M b^2} \qquad \underline{I_z = \frac{1}{12} M (a^2+b^2)} \qquad \underline{I_{xy} = I_{yz} = I_{zx} = 0}$$

$$\underset{\sim}{\omega}_{plate} = \omega\, \hat{\underset{\sim}{e}}_X = \omega \cos\alpha\, \hat{\underset{\sim}{e}}_x - \omega \sin\alpha\, \hat{\underset{\sim}{e}}_y \longrightarrow \begin{cases} \omega_x = \omega\cos\alpha \\ \omega_y = -\omega\sin\alpha \end{cases} \quad \omega_z = 0$$

$$\underset{\sim}{H}_c = [I]_c\, \underset{\sim}{\omega} \quad \Longrightarrow \quad \underset{\sim}{H}_c = I_x \omega_x\, \hat{\underset{\sim}{e}}_x + I_y \omega_y\, \hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{\tau}_c = \frac{d\underset{\sim}{H}_c}{dt}$$

$$\underset{\sim}{\omega} \times \hat{\underset{\sim}{e}}_x = \omega \sin\alpha\, \hat{\underset{\sim}{e}}_z$$

$$\frac{d\underset{\sim}{H}_c}{dt} = I_x \omega_x \frac{d\hat{\underset{\sim}{e}}_x}{dt} + I_y \omega_y \frac{d\hat{\underset{\sim}{e}}_y}{dt} \qquad \underset{\sim}{\omega} \times \hat{\underset{\sim}{e}}_y = \omega \cos\alpha\, \hat{\underset{\sim}{e}}_z$$

$$= \left(\frac{1}{12} M a^2 \omega^2 \cos\alpha \sin\alpha - \frac{1}{12} M b^2 \omega^2 \sin\alpha \cos\alpha\right) \hat{\underset{\sim}{e}}_z = \frac{1}{12} M \omega^2 \sin\alpha \cos\alpha\, (a^2 - b^2)\, \hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\tau}_c = F\sqrt{a^2+b^2}\; \hat{\underset{\sim}{e}}_z$$

$$\therefore \quad F\sqrt{a^2+b^2} = \frac{1}{12} M\omega^2 \frac{ab}{a^2+b^2}(a^2-b^2)$$

$$\Longrightarrow \quad \boxed{F = \frac{1}{12} M\omega^2 ab \frac{a^2-b^2}{(a^2+b^2)^{3/2}}}$$

Note that force F rotates about X axis and is always in xy plane.

(b)

$$KE. = \frac{1}{2}\{\omega\}^t [I]_c \{\omega\} + \cancelto{0}{\frac{1}{2} M \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c}$$

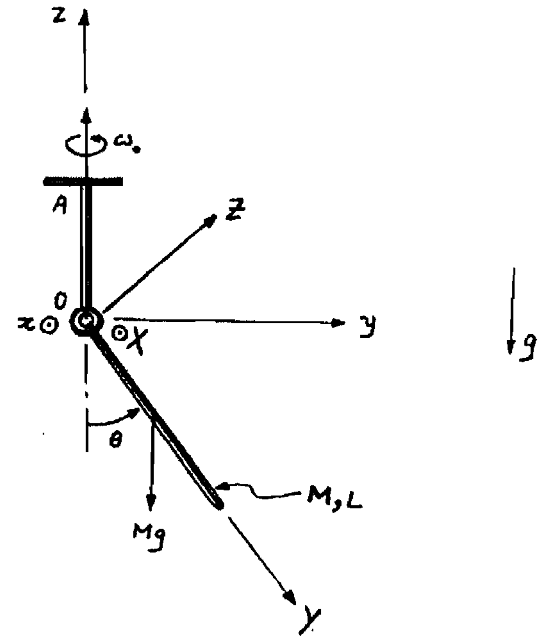
$$= \frac{1}{2}\left(I_x \omega_x^2 + I_y \omega_y^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{12} Ma^2\omega^2 \cos^2\alpha + \frac{1}{12} Mb^2\omega^2 \sin^2\alpha\right)$$

$$= \boxed{\frac{1}{12} M\omega^2 \frac{a^2b^2}{a^2+b^2}}$$

kinetic energy of the rotating plate

Problem 4



xyz coordinate system rotates about z with ω_0 so that the rod is always in yz plane.

XYZ coordinate system is fixed to the rod.

(a)

Point O is stationary, so we have

$$\underset{\sim}{H}_0 = [I]_0\, \underset{\sim}{\omega}_{rod} \quad , \qquad \underset{\sim}{\tau}_0 = \frac{d}{dt}\underset{\sim}{H}_0$$

$$\underset{\sim}{\omega}_{rod} = \omega_0 \underset{\sim}{\hat{e}}_z + \dot{\theta}\underset{\sim}{\hat{e}}_x = \omega_0\left(-\cos\theta\, \underset{\sim}{\hat{e}}_Y + \sin\theta\, \underset{\sim}{\hat{e}}_Z\right) + \dot{\theta}\,\underset{\sim}{\hat{e}}_X$$

To find $[I]_0$ for the rod,

$$I_X = \int \rho\, dV\, (Y^2 + Z^2) = \int_0^L \frac{M}{L}\, dY\, Y^2 = M\frac{L^2}{3}$$

$$I_Y = \int \rho\, dV\, (X^2 + Z^2) = 0 \qquad (X \approx 0 ,\quad Z \approx 0)$$

$$I_Z = \int \rho\, dV (Y^2 + X^2) = \int_0^L \frac{M}{L}\, dY\, Y^2 = M\frac{L^2}{3}$$

$$I_{XY} = I_{YZ} = I_{ZX} = 0$$

$$\therefore \quad [I]_0 = \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix} \frac{ML^2}{3}$$

$$\underset{\sim}{H}_0 = [I]_0 \begin{Bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{Bmatrix} = \frac{ML^2}{3}\dot{\theta}\,\underset{\sim}{\hat{e}}_X + \frac{ML^2}{3}\omega_0 \sin\theta\, \underset{\sim}{\hat{e}}_Z = \frac{ML^2}{3}\left[\dot{\theta}\,\underset{\sim}{\hat{e}}_x + \omega_0 \sin\theta\left(\cos\theta\, \underset{\sim}{\hat{e}}_y + \sin\theta\, \underset{\sim}{\hat{e}}_z\right)\right]$$

$$\frac{d}{dt}\underset{\sim}{H}_0 = \frac{ML^2}{3}\left[\ddot{\theta}\,\underset{\sim}{\hat{e}}_x + \dot{\theta}\underbrace{\frac{d\underset{\sim}{\hat{e}}_x}{dt}}_{\omega_0 \underset{\sim}{\hat{e}}_z \times \underset{\sim}{\hat{e}}_x = \omega_0 \underset{\sim}{\hat{e}}_y} + \omega_0\dot{\theta}\cos 2\theta\, \underset{\sim}{\hat{e}}_y + \omega_0 \frac{\sin 2\theta}{2}\underbrace{\frac{d\underset{\sim}{\hat{e}}_y}{dt}}_{\omega_0 \underset{\sim}{\hat{e}}_z \times \underset{\sim}{\hat{e}}_y = -\omega_0 \underset{\sim}{\hat{e}}_x} + \omega_0\dot{\theta}\sin 2\theta\, \underset{\sim}{\hat{e}}_z\right]$$

Problem 4

$$\underset{\sim}{\tau}_0 = -Mg\frac{L}{2}\sin\theta\,\hat{\underset{\sim}{e}}_x + M_y\,\hat{\underset{\sim}{e}}_y + M_z\,\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\tau}_0 = \frac{d}{dt}\underset{\sim}{H}_0$$

$$\therefore \quad -Mg\frac{L}{2}\sin\theta = \frac{ML^2}{3}\left(\ddot{\theta} - \omega_0^2\sin\theta\cos\theta\right)$$

$$\Longrightarrow \quad \underline{L\ddot{\theta} + \left(\frac{3g}{2} - L\omega_0^2\cos\theta\right)\sin\theta = 0} \qquad \text{equation of motion for } \theta(t)$$

(b)

For stationary angle θ, $\dot{\theta} = \ddot{\theta} = 0$

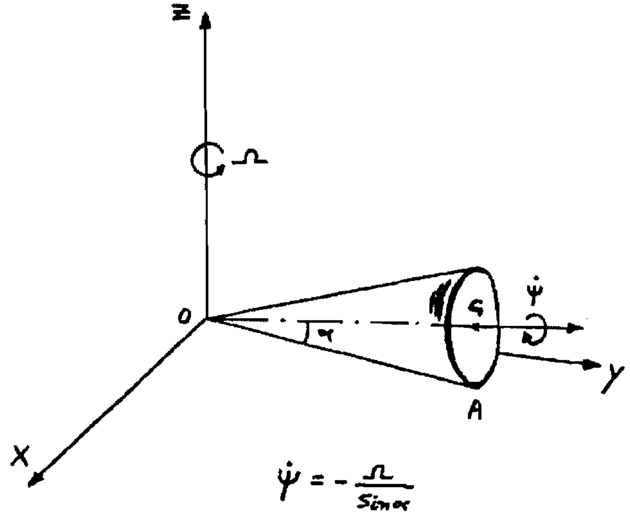
$$\ddot{\theta} = 0 \Longrightarrow \left(\frac{3g}{2} - L\omega_0^2\cos\theta\right)\sin\theta = 0 \Longrightarrow \begin{cases} \theta_0 = 0 \\ \theta_0 = \cos^{-1}\left(\frac{3g}{2L\omega_0^2}\right), & \frac{3g}{2L\omega_0^2} < 1 \end{cases}$$

$$\text{Stationary angles } \theta_0 : \begin{cases} \theta_0 = 0, & \omega_0^2 < \frac{3g}{2L} \\ \theta_0 = \cos^{-1}\left(\frac{3g}{2L\omega_0^2}\right) \;\&\; \theta_0 = 0, & \omega_0^2 > \frac{3g}{2L} \end{cases}$$

It can be shown that $\theta_0 = 0$ is unstable when $\omega_0^2 > \frac{3g}{2L}$.

Problem 5

xyz rotates with Ω about Z axis.



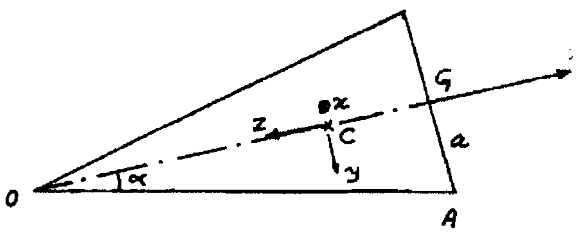
$\dot\psi = -\dfrac{\Omega}{\sin\alpha}$

$$\underset{\sim}{\omega}_{cone} = -\Omega \cot\alpha \;\underset{\sim}{\hat e}_{OA}$$
$$= -\Omega \cot\alpha \left(-\cos\alpha\, \underset{\sim}{\hat e}_z + \sin\alpha\, \underset{\sim}{\hat e}_y\right)$$
$$= -\Omega \cos\alpha\, \underset{\sim}{\hat e}_y + \Omega \frac{\cos^2\alpha}{\sin\alpha}\, \underset{\sim}{\hat e}_z$$

(a)

Angular momentum of the cone about the tip 0 : $\underset{\sim}{H}_0 = [I]_0\, \underset{\sim}{\omega}$

$$[I]_0 = \begin{bmatrix} I_1 & 0 & 0 \\ & I_1 & 0 \\ & & I_3 \end{bmatrix}$$



$$\underset{\sim}{\omega} = \begin{Bmatrix} \omega_x = 0 \\ \omega_y = -\Omega\cos\alpha \\ \omega_z = \Omega \dfrac{\cos^2\alpha}{\sin\alpha} \end{Bmatrix}$$

$a = \sqrt{\dfrac{10}{3}\dfrac{I_3}{M}}$

$$\therefore \quad \boxed{\underset{\sim}{H}_0 = [I]_0\, \underset{\sim}{\omega} = -I_1 \Omega \cos\alpha\, \underset{\sim}{\hat e}_y + I_3 \Omega \frac{\cos^2\alpha}{\sin\alpha}\, \underset{\sim}{\hat e}_z}$$

(b)

$$\underset{\sim}{P} = M \underset{\sim}{V}_C = M\Omega \frac{3}{4} a \frac{\cos^2\alpha}{\sin\alpha}\, \underset{\sim}{\hat e}_x$$

$$\underset{\sim}{F} = \frac{d\underset{\sim}{P}}{dt} = \frac{3}{4} M\Omega a \frac{\cos^2\alpha}{\sin\alpha} \frac{d\underset{\sim}{\hat e}_x}{dt}$$

$$\Omega\, \underset{\sim}{\hat e}_Z \times \underset{\sim}{\hat e}_x = \Omega\left(-\cos\alpha\, \underset{\sim}{\hat e}_y - \sin\alpha\, \underset{\sim}{\hat e}_z\right) \times \underset{\sim}{\hat e}_x$$
$$= \Omega\left(\cos\alpha\, \underset{\sim}{\hat e}_z - \sin\alpha\, \underset{\sim}{\hat e}_y\right) = \Omega\, \underset{\sim}{\hat e}_{AO}$$

$$\boxed{\underset{\sim}{F} = \frac{3}{4} M\Omega^2 a \frac{\cos^2\alpha}{\sin\alpha}\, \underset{\sim}{\hat e}_{AO}}$$ total required force

Problem 5

(b)

$$\underset{\sim}{\tau}_o = \frac{d\underset{\sim}{H}_o}{dt} \qquad (\underset{\sim}{v}_o = 0)$$

$$\frac{d\underset{\sim}{H}_o}{dt} = -I_1 \Omega \cos\alpha \frac{d\hat{\underset{\sim}{e}}_y}{dt} + I_3 \Omega \frac{\cos^2\alpha}{\sin\alpha} \frac{d\hat{\underset{\sim}{e}}_z}{dt}$$

$$\frac{d\hat{\underset{\sim}{e}}_y}{dt} = \Omega \hat{\underset{\sim}{e}}_z \times \hat{\underset{\sim}{e}}_y = \Omega \left(-\cos\alpha\, \hat{\underset{\sim}{e}}_y - \sin\alpha\, \hat{\underset{\sim}{e}}_z\right) \times \hat{\underset{\sim}{e}}_y = \Omega \sin\alpha\, \hat{\underset{\sim}{e}}_x$$

$$\frac{d\hat{\underset{\sim}{e}}_z}{dt} = \Omega \hat{\underset{\sim}{e}}_z \times \hat{\underset{\sim}{e}}_z = \Omega \left(-\cos\alpha\, \hat{\underset{\sim}{e}}_y - \sin\alpha\, \hat{\underset{\sim}{e}}_z\right) \times \hat{\underset{\sim}{e}}_z = -\Omega \cos\alpha\, \hat{\underset{\sim}{e}}_x$$

$$\therefore \quad \boxed{\underset{\sim}{\tau}_o = -\Omega^2 \left(I_1 \sin\alpha \cos\alpha + I_3 \frac{\cos^3\alpha}{\sin\alpha}\right) \hat{\underset{\sim}{e}}_x}$$

total required torque about point o