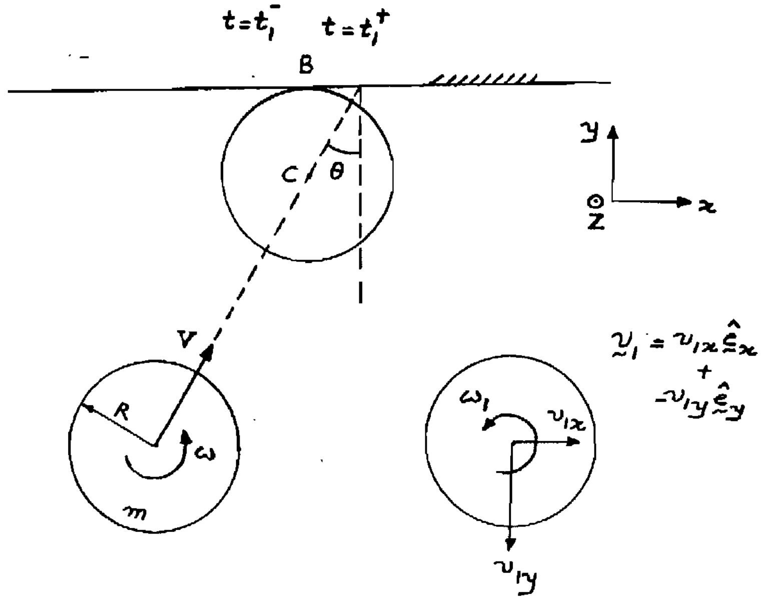
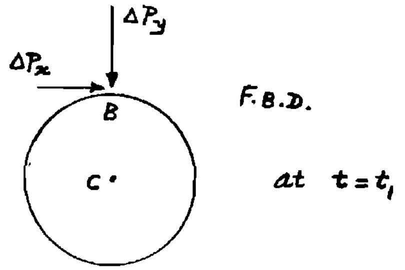
Problem Set No. 5

Problem 1



Assume the disk collides with the wall at point B at $t=t_1$.

An impulse acts on the disk at $t=t_1$ $(\Delta P_x, \Delta P_y)$.



Collision in the normal direction (y) is elastic so the magnitude of the velocity in the normal direction is conserved:

$$v_{1y}\Big|_{t=t_1^+} = v_y\Big|_{t=t_1^-} \quad \Rightarrow \quad v_{1y}\Big|_{t=t_1^+} = V\cos\theta$$

$$\therefore \quad v_{1y} = V\cos\theta$$

x component

No slip occurs at the wall. $\Rightarrow \quad v_{Bx}\Big|_{t=t_1^+} = 0 \quad \Rightarrow \quad \Big[v_{Cx} + (\omega \times CB)_x\Big]_{t=t_1^+} = 0$

$$\Rightarrow \quad v_{1x} - \omega_1 R = 0 \quad \Rightarrow \quad v_{1x} = R\omega_1$$

Angular momentum about point B: $\quad \tau_B = \frac{d}{dt}H_B + \cancel{v_B}^{\,0} \times P$

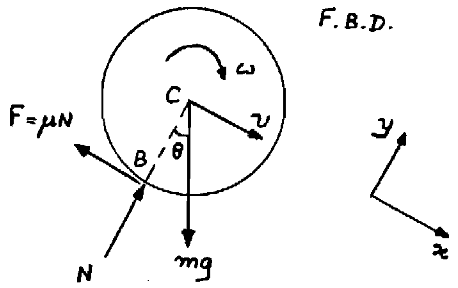
$$\tau_B = 0 \longrightarrow \frac{d}{dt}H_B = 0 \longrightarrow H_B\Big|_{t=t_1^-} = H_B\Big|_{t=t_1^+} \qquad (H_B = H_C + BC \times P)$$

$$\longrightarrow \frac{1}{2}mR^2\omega + mVR\sin\theta = \frac{1}{2}mR^2\omega_1 + mRv_{1x} \xRightarrow{\omega_1 = v_{1x}/R} v_{1x} = \frac{2}{3}V\sin\theta + \frac{1}{3}R\omega$$

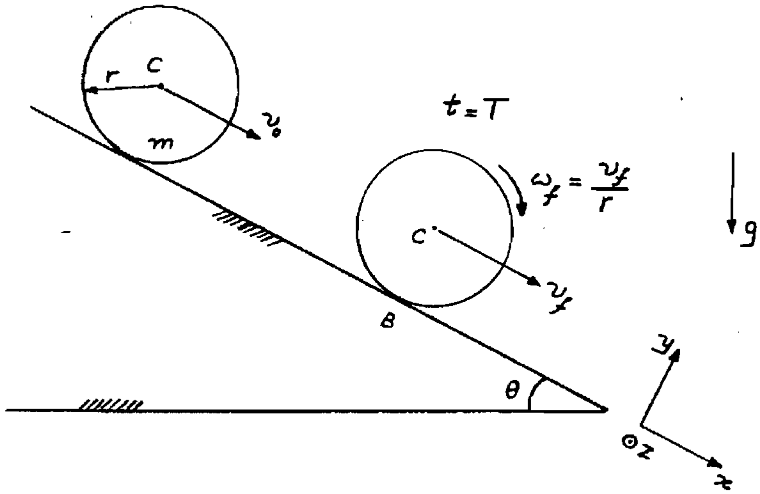
$$v_1 = \sqrt{v_{1x}^2 + v_{1y}^2} = \sqrt{\left(\frac{2}{3}V\sin\theta + \frac{1}{3}R\omega\right)^2 + (V\cos\theta)^2}$$ velocity of the center of the disk

Problem 2

$0 < t < T$



$t = 0$



End of the period of slipping $t = T$.

Linear mom. : $m \frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{F}$

in x direction : $m \frac{dv}{dt} = mg \sin\theta - F$ $\qquad$ ($F = \mu N$ during the period of slipping)

in y direction : $0 = mg \cos\theta - N \longrightarrow N = mg \cos\theta$

$$\therefore \quad m \frac{dv}{dt} = mg\,(\sin\theta - \mu \cos\theta) \longrightarrow \int_0^T dv = \int_0^T g\,(\sin\theta - \mu\cos\theta)\,dt$$

$$\Longrightarrow \quad \underline{v_f - v_0 = gT\,(\sin\theta - \mu\cos\theta)}$$

Ang. mom. about point C : $\underset{\sim}{\tau}_C = \frac{d}{dt}\underset{\sim}{H}_C$

$$\left.\begin{aligned} \underset{\sim}{\tau}_C &= -\mu N r\,\hat{\underset{\sim}{e}}_z = -\mu mg\cos\theta\, r\,\hat{\underset{\sim}{e}}_z \\ \frac{d}{dt}\underset{\sim}{H}_C &= -I_C \frac{d\omega}{dt}\,\hat{\underset{\sim}{e}}_z = -\frac{2}{5} m r^2 \frac{d\omega}{dt}\,\hat{\underset{\sim}{e}}_z \end{aligned}\right\} \Longrightarrow \mu g \cos\theta = \frac{2}{5} r \frac{d\omega}{dt}$$

$$\Longrightarrow \int_0^T d\omega = \int_0^T \frac{5}{2}\frac{\mu}{r} g \cos\theta\,dt \longrightarrow \underline{\omega_f - 0 = \frac{5}{2}\mu g \cos\theta \frac{T}{r}}$$

No slip at $t = T \Longrightarrow \underline{r\omega_f = v_f}$

Problem 2

$$\therefore \quad v_f = \frac{5}{2}\mu g \cos\theta \, T$$

$$v_f - v_0 = g T (\sin\theta - \mu \cos\theta)$$

$$\Longrightarrow \quad \boxed{T = \frac{2 v_0}{g(7\mu\cos\theta - 2\sin\theta)}}$$

time duration of slipping

$$\boxed{v_f = \frac{5\mu\cos\theta \, v_0}{7\mu\cos\theta - 2\sin\theta}}$$

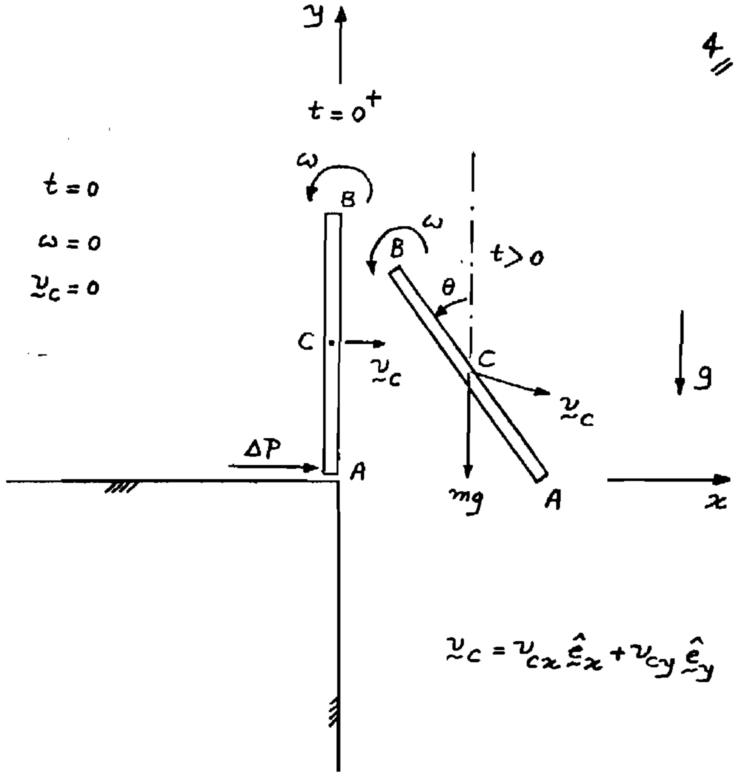
velocity of the center of mass C at the end of the period of slipping

Note that $7\mu\cos\theta - 2\sin\theta$ has to be positive :

$$\tan\theta < 3.5\mu$$

Problem 3

$AB = 2a$



Horizontal impulse ΔP at $t=0$.

During the impulse period, other forces do not have enough time to act:

$$\underset{\sim}{v}_c = v_{cx}\,\underset{\sim}{\hat{e}}_x + v_{cy}\,\underset{\sim}{\hat{e}}_y$$

Linear momentum: $\quad m\dfrac{d\underset{\sim}{v}_c}{dt} = \underset{\sim}{F}$

$$m\,d\underset{\sim}{v}_c = \underset{\sim}{F}\,dt$$

$$m\left(\underset{\sim}{v}_c\Big|_{t=0^+} - 0\right) = \int_0^{0^+} \underset{\sim}{F}\,dt = \Delta\underset{\sim}{P} = \Delta P\,\underset{\sim}{\hat{e}}_x$$

$$\Longrightarrow \quad \underset{\sim}{v}_c\Big|_{t=0^+} = \frac{\Delta P}{m}\,\underset{\sim}{\hat{e}}_x \quad \longrightarrow \quad \begin{cases} v_{cx} = \dfrac{\Delta P}{m} \\ v_{cy} = 0 \end{cases}, \quad \text{at} \quad t = 0^+$$

Ang. mom. about C: $\quad \underset{\sim}{\tau}_c = \dfrac{d}{dt}\underset{\sim}{H}_c = I_c\dfrac{d\omega}{dt}\,\underset{\sim}{\hat{e}}_z$

$$\underset{\sim}{\tau}_c\,dt = I_c\,d\omega\,\underset{\sim}{\hat{e}}_z \quad \Longrightarrow \quad \int_0^{0^+}\underset{\sim}{\tau}_c\,dt = I_c\left(\omega\Big|_{t=0^+} - 0\right)\underset{\sim}{\hat{e}}_z$$

$$\Longrightarrow \quad \Delta P\,a = \frac{1}{12}m(2a)^2\left[\omega\Big|_{t=0^+} - 0\right] \quad \longrightarrow \quad \omega\Big|_{t=0^+} = \frac{3\Delta P}{ma}$$

$\underset{\sim}{v}_c\Big|_{t=0^+}$ and $\omega\Big|_{t=0^+}$ are the initial conditions for the next stage of the motion which is free fall:

$$\underset{\sim}{\tau}_c = \frac{d}{dt}\underset{\sim}{H}_c\,, \quad \underset{\sim}{\tau}_c = 0 \quad \Longrightarrow \quad I_c\,\omega = \text{const.} \quad \longrightarrow \quad \omega = \text{Const.} = \omega\Big|_{t=0^+} = \frac{3\Delta P}{ma}$$

$$m\frac{d\underset{\sim}{v}_c}{dt} = \underset{\sim}{F} = -mg\,\underset{\sim}{\hat{e}}_y$$

Problem 3

$$\therefore \begin{cases} m\dfrac{dv_{cx}}{dt} = 0 \longrightarrow v_{cx} = \text{const.} = v_{cx}\big|_{t=0^+} = \dfrac{\Delta P}{m} \\ m\dfrac{dv_{cy}}{dt} = -mg \longrightarrow v_{cy} = -gt + \overbrace{v_{cy}\big|_{t=0^+}}^{0} = -gt \end{cases}$$

$$\begin{cases} v_{cx} = \dfrac{dx_c}{dt} = \dfrac{\Delta P}{m} \longrightarrow x_c = \dfrac{\Delta P}{m}t + \overbrace{x_c\big|_{t=0}}^{0} = \dfrac{\Delta P}{m}t \\ v_{cy} = \dfrac{dy_c}{dt} = -gt \longrightarrow y_c = -g\dfrac{t^2}{2} + \overbrace{y_c\big|_{t=0}}^{a} = -g\dfrac{t^2}{2} + a \end{cases}$$

$$\omega = \frac{d\theta}{dt} = \frac{3\Delta P}{ma} \longrightarrow \theta = \frac{3\Delta P}{ma}t + \overbrace{\theta\big|_{t=0}}^{0} = \frac{3\Delta P}{ma}t$$

$$\begin{cases} x_B = x_C - a\sin\theta = \dfrac{\Delta P}{m}t - a\sin\left(\dfrac{3\Delta P}{ma}t\right) \\ y_B = y_C + a\cos\theta = -g\dfrac{t^2}{2} + a\left[1+\cos\left(\dfrac{3\Delta P}{ma}t\right)\right] \end{cases}$$

Point B clips the edge of the table $(x=0,\ y=0)$:

$$\begin{cases} x_B = 0 = \dfrac{\Delta P}{m}t - a\sin\left(\dfrac{3\Delta P}{ma}t\right) = 0 \xrightarrow{X = \frac{\Delta P t}{ma}} X - \sin(3X) = 0 \longrightarrow X = 0.76 = \dfrac{\Delta P t}{ma} \\ y_B = 0 = -g\dfrac{t^2}{2} + a\left[1+\cos\left(\dfrac{3\Delta P}{ma}t\right)\right] = 0 \end{cases}$$

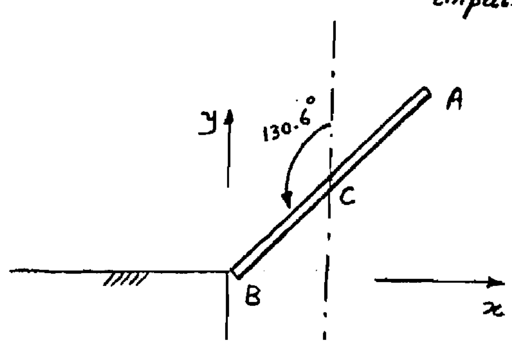
$$-g\frac{t^2}{2} + a\left[1+\cos(3X)\right] = 0 \longrightarrow g\frac{t^2}{2} = a(1-0.65) \longrightarrow \underline{t = 0.84\sqrt{\frac{a}{g}}}$$

$$\Delta P t = 0.76\, ma \longrightarrow \Delta P = \frac{0.76\, ma}{0.84\sqrt{\frac{a}{g}}} = \boxed{0.91\, m\sqrt{ga}}$$

value of horizontal impulse

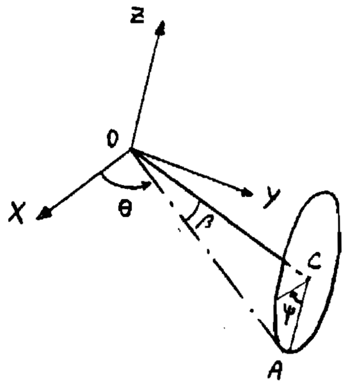
At this instant :

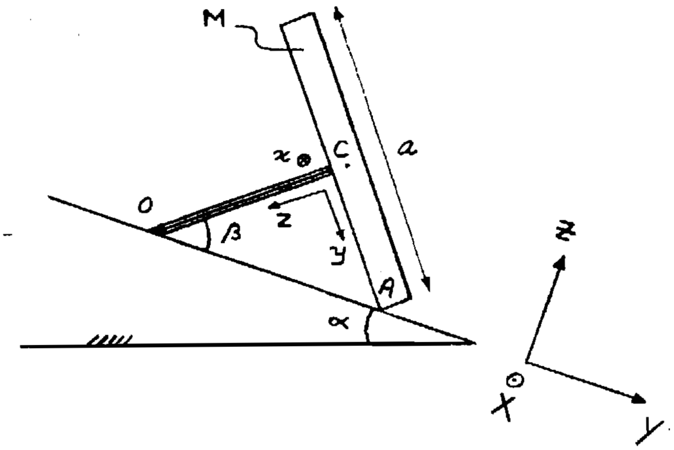
$$\begin{cases} \theta = \dfrac{3\Delta P}{ma}t = 3(0.76) = 2.28 = \underline{130.6^\circ} \\ x_B = y_B = 0 \end{cases}$$



Position of the pencil at this instant.

Problem 4





$$I_1 = I_2 = I_x = I_y = \frac{Ma^2}{16}$$

$$I_3 = I_z = \frac{Ma^2}{8}$$

β is the angle that OC makes with the slope.

$$\underset{\sim}{\omega}_{disk} = \dot{\theta}\,\hat{\underset{\sim}{e}}_Z + \dot{\psi}\,\hat{\underset{\sim}{e}}_z \qquad \hat{\underset{\sim}{e}}_z = -\cos\beta\cos\theta\,\hat{\underset{\sim}{e}}_X - \cos\beta\sin\theta\,\hat{\underset{\sim}{e}}_Y - \sin\beta\,\hat{\underset{\sim}{e}}_Z$$

$$\underset{\sim}{\omega}_{disk} = -\dot{\psi}\cos\beta\cos\theta\,\hat{\underset{\sim}{e}}_X - \dot{\psi}\cos\beta\sin\theta\,\hat{\underset{\sim}{e}}_Y + (\dot{\theta} - \dot{\psi}\sin\beta)\,\hat{\underset{\sim}{e}}_Z$$

No slip : $\quad \underset{\sim}{v}_A\big|_{disk} = \underset{\sim}{0}$

$$\underset{\sim}{v}_A\big|_{disk} = \overset{\underset{\sim}{0}}{\underset{\sim}{v}_0} + \underset{\sim}{\omega}\big|_{disk} \times \underset{\sim}{OA} = \underset{\sim}{0} \qquad \underset{\sim}{OA} = \frac{a}{2\sin\beta}\left(\cos\theta\,\hat{\underset{\sim}{e}}_X + \sin\theta\,\hat{\underset{\sim}{e}}_Y\right)$$

$$\underset{\sim}{\omega}\big|_{disk} \times \underset{\sim}{OA} = \underset{\sim}{0} \implies \dot{\psi} = \frac{\dot{\theta}}{\sin\beta} \longrightarrow \underset{\sim}{\omega}_{disk} = -\dot{\theta}\cot\beta\,\hat{\underset{\sim}{e}}_{OA}$$

$$T = \frac{1}{2}M\underset{\sim}{v}_c\cdot\underset{\sim}{v}_c + \frac{1}{2}\{\omega\}^t[I]_c\{\omega\} = \frac{1}{2}M\underset{\sim}{v}_c\cdot\underset{\sim}{v}_c + \frac{1}{2}\left(I_1\omega_x^2 + I_1\omega_y^2 + I_3\omega_z^2\right)$$

$$\underset{\sim}{v}_c = \dot{\theta}\,\hat{\underset{\sim}{e}}_Z \times \underset{\sim}{OC} = \dot{\theta}\,\frac{a}{2\tan\beta}\cos\beta\,\hat{\underset{\sim}{e}}_x \longrightarrow \underset{\sim}{v}_c\cdot\underset{\sim}{v}_c = \dot{\theta}^2 a^2\,\frac{\cos^4\beta}{4\sin^2\beta}$$

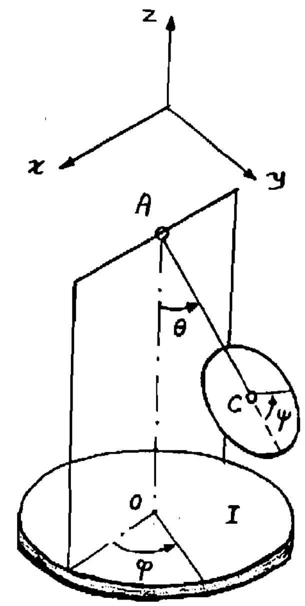
$$\underset{\sim}{\omega}_{disk} = -\dot{\theta}\cot\beta\,\hat{\underset{\sim}{e}}_{OA} = -\dot{\theta}\cot\beta\left(-\cos\beta\,\hat{\underset{\sim}{e}}_z + \sin\beta\,\hat{\underset{\sim}{e}}_y\right) \longrightarrow \omega_x = 0\,,\quad \omega_y = -\dot{\theta}\cos\beta\,,\quad \omega_z = \dot{\theta}\,\frac{\cos^2\beta}{\sin\beta}$$

$$\therefore\ T = \frac{1}{8}Ma^2\dot{\theta}^2\,\frac{\cos^4\beta}{\sin^2\beta} + \frac{1}{2}I_1\dot{\theta}^2\cos^2\beta + \frac{1}{2}I_3\dot{\theta}^2\,\frac{\cos^4\beta}{\sin^2\beta} = \underline{\frac{\cos^2\beta}{32}\left(1 + 6\cot^2\beta\right)Ma^2\dot{\theta}^2}$$

kinetic energy of the disk

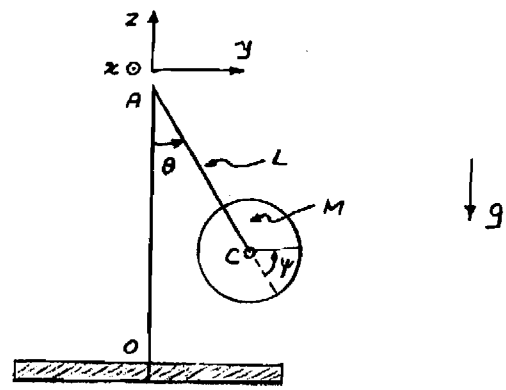
Problem 5

(a)



$\underline{\varphi,\ \theta \text{ and } \psi}$ are an independent and complete set of generalized coordinates.

(b)



$$\underline{P.E. = -MgL\cos\theta}$$

Potential energy of the system

$$K.E. = \frac{1}{2}\{\omega\}^t_{table}[I]_O\{\omega\}_{table}$$

$$+\frac{1}{2}M\,\underset{\sim}{v}_{c\,disk}\cdot\underset{\sim}{v}_{c\,disk} + \frac{1}{2}\{\omega\}^t_{disk}[I]_C\{\omega\}_{disk}$$

$$\underset{\sim}{v}_c\big|_{disk} = \underset{\sim}{\omega}_{AC}\times\underset{\sim}{AC} = \left[\dot\theta\,\hat{\underset{\sim}{e}}_x + \dot\varphi\,\hat{\underset{\sim}{e}}_z\right]\times\left[L\sin\theta\,\hat{\underset{\sim}{e}}_y - L\cos\theta\,\hat{\underset{\sim}{e}}_z\right] = -\dot\varphi L\sin\theta\,\hat{\underset{\sim}{e}}_x + \dot\theta L\cos\theta\,\hat{\underset{\sim}{e}}_y + \dot\theta L\sin\theta\,\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\omega}_{table} = \dot\varphi\,\hat{\underset{\sim}{e}}_z \longrightarrow \{\omega\}_{table} = \begin{Bmatrix}0\\0\\\dot\varphi\end{Bmatrix}$$

$$\underset{\sim}{\omega}_{disk} = \dot\psi\,\hat{\underset{\sim}{e}}_x + \dot\theta\,\hat{\underset{\sim}{e}}_x + \dot\varphi\,\hat{\underset{\sim}{e}}_z \longrightarrow \{\omega\}_{disk} = \begin{Bmatrix}\dot\psi+\dot\theta\\0\\\dot\varphi\end{Bmatrix}$$

$$K.E. = \frac{1}{2}\begin{Bmatrix}0\\0\\\dot\varphi\end{Bmatrix}^t\begin{bmatrix}I_0 & 0 & 0\\ & I_0 & 0\\ & & I\end{bmatrix}\begin{Bmatrix}0\\0\\\dot\varphi\end{Bmatrix} + \frac{1}{2}Mv^2_{c\,disk} + \frac{1}{2}\begin{Bmatrix}\dot\psi+\dot\theta\\0\\\dot\varphi\end{Bmatrix}^t\begin{bmatrix}I_3 & 0 & 0\\ & I_1 & 0\\ & & I_1\end{bmatrix}\begin{Bmatrix}\dot\psi+\dot\theta\\0\\\dot\varphi\end{Bmatrix}$$

$$\underline{K.E. = \frac{1}{2}I\dot\varphi^2 + \frac{1}{2}M\left(\dot\varphi^2L^2\sin^2\theta + \dot\theta^2L^2\right) + \frac{1}{2}I_3\left(\dot\psi+\dot\theta\right)^2 + \frac{1}{2}I_1\dot\varphi^2}$$

Kinetic energy of the system