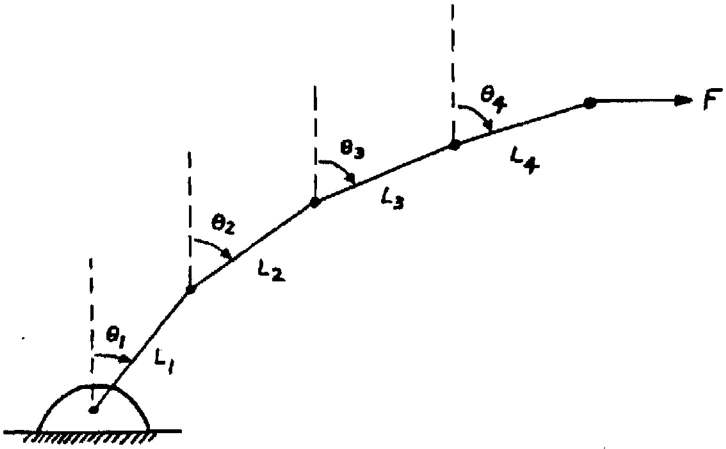
2.032 Dynamics

Problem Set No. 6

Problem 1



$\xi_1 = \theta_1$

$\xi_2 = \theta_2$

$\xi_3 = \theta_3$

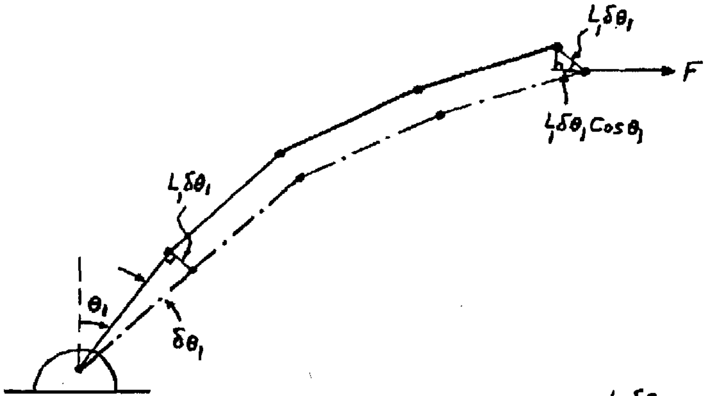
$\xi_4 = \theta_4$

$\Xi_1 = ?$ $\Xi_2 = ?$

To find Ξ_1,

Freeze $\theta_2, \theta_3, \theta_4$, and

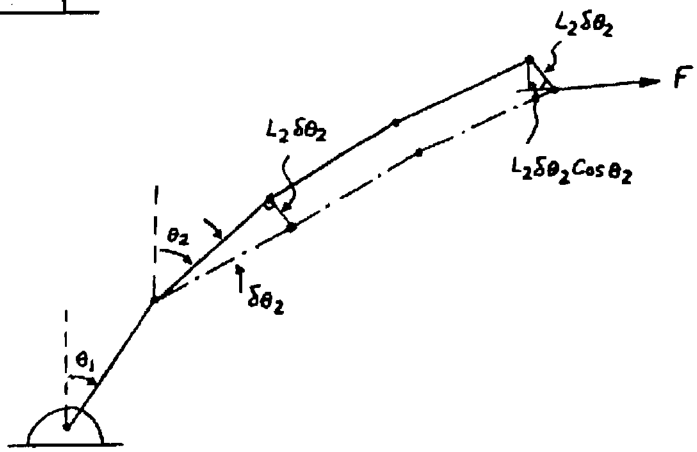
vary $\theta_1 \longrightarrow \theta_1 + \delta\theta_1$



$$\delta W_1 = F\,(L_1\delta\theta_1)\cos\theta_1$$

$$= (F L_1 \cos\theta_1)\,\delta\theta_1$$

$$\therefore \quad \boxed{\Xi_1 = F L_1 \cos\theta_1}$$



To find Ξ_2,

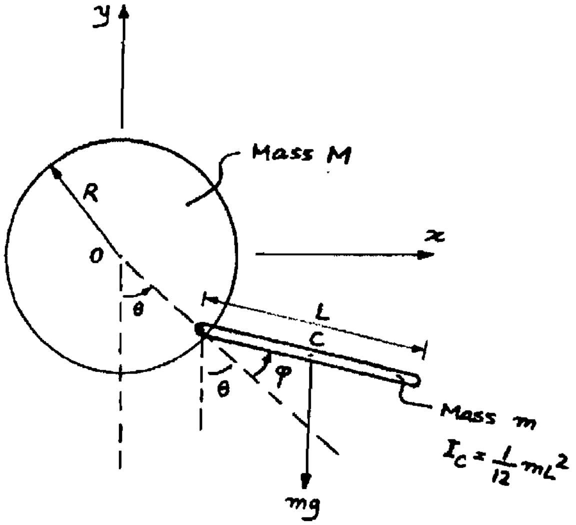
Freeze $\theta_1, \theta_3, \theta_4$, and

vary $\theta_2 \longrightarrow \theta_2 + \delta\theta_2$

$$\delta W_2 = F\,(L_2\delta\theta_2)\cos\theta_2 = (F L_2 \cos\theta_2)\,\delta\theta_2 \quad \Longrightarrow \quad \boxed{\Xi_2 = F L_2 \cos\theta_2}$$

Problem 2

a) $\xi_1 = \theta$ and $\xi_2 = \varphi$ are complete and independent set of generalized coordinates.



b)

$$\underset{\sim}{\omega}_M = \underset{\sim}{\omega}\big|_{flywheel} = \dot\theta \,\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\omega}_m = \underset{\sim}{\omega}\big|_{rod} = (\dot\theta + \dot\varphi)\,\hat{\underset{\sim}{e}}_z$$

$$\begin{cases} x_c = R\sin\theta + \frac{L}{2}\sin(\theta+\varphi) \\ y_c = -R\cos\theta - \frac{L}{2}\cos(\theta+\varphi) \end{cases}$$

$$\Longrightarrow \quad \underset{\sim}{v}_c = \left[R\cos\theta\,\dot\theta + \frac{L}{2}\cos(\theta+\varphi)\,(\dot\theta+\dot\varphi)\right]\hat{\underset{\sim}{e}}_x + \left[R\sin\theta\,\dot\theta + \frac{L}{2}\sin(\theta+\varphi)\,(\dot\theta+\dot\varphi)\right]\hat{\underset{\sim}{e}}_y$$

Construct Lagrangian:

$$\mathcal{L} = T - V$$

$$T = T_M + T_m$$

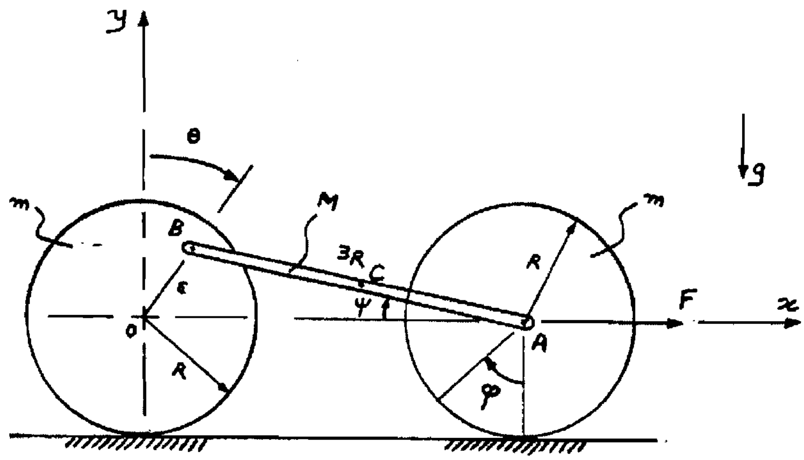
$$T_M = \frac{1}{2} I_0\, \underset{\sim}{\omega}_M \cdot \underset{\sim}{\omega}_M = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\dot\theta^2 = \frac{1}{4}MR^2\dot\theta^2$$

$$T_m = \frac{1}{2} m\, \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c + \frac{1}{2} I_c\, \underset{\sim}{\omega}_m \cdot \underset{\sim}{\omega}_m = \frac{1}{2} m\left[R^2\dot\theta^2 + \frac{L^2}{4}(\dot\theta+\dot\varphi)^2 + LR\dot\theta\,(\dot\theta+\dot\varphi)\cos\varphi\right]$$
$$+ \frac{1}{2}\left(\frac{1}{12}mL^2\right)(\dot\theta+\dot\varphi)^2$$

$$V = mg\,y_c = -mg\left[R\cos\theta - \frac{L}{2}\cos(\theta+\varphi)\right] \qquad (y_0 = 0)$$

$$\therefore \quad \mathcal{L} = \left(\frac{M}{4} + \frac{m}{2}\right)R^2\dot\theta^2 + \frac{1}{6}mL^2(\dot\theta+\dot\varphi)^2 + \frac{1}{2}mLR\dot\theta(\dot\theta+\dot\varphi)\cos\varphi + mgR\cos\theta + mg\frac{L}{2}\cos(\theta+\varphi)$$

Problem 3

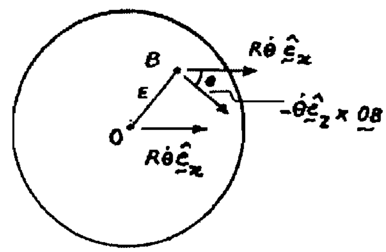


$$\underset{\sim}{\omega}\Big|_{\text{left cyl.}} = -\dot{\theta}\,\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\omega}\Big|_{\text{right cyl.}} = -\dot{\varphi}\,\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\omega}\Big|_{\text{rod}} = -\dot{\psi}\,\hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{v}_A = R\dot{\varphi}\,\hat{\underset{\sim}{e}}_x$$

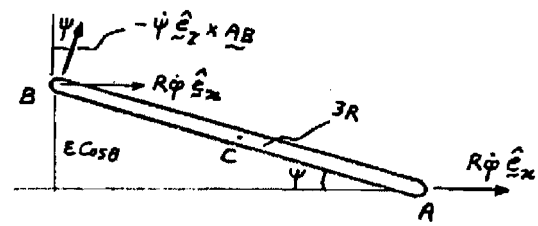


$$\underset{\sim}{v}_B = \underset{\sim}{v}_O + (-\dot{\theta}\,\hat{\underset{\sim}{e}}_z) \times \underset{\sim}{OB}$$

$$= R\dot{\theta}\,\hat{\underset{\sim}{e}}_x + \dot{\theta}\varepsilon\left(\cos\theta\,\hat{\underset{\sim}{e}}_x - \sin\theta\,\hat{\underset{\sim}{e}}_y\right)$$

$$= (R + \varepsilon\cos\theta)\dot{\theta}\,\hat{\underset{\sim}{e}}_x - \varepsilon\sin\theta\,\dot{\theta}\,\hat{\underset{\sim}{e}}_y$$

(point B on the left cylinder)



$$\underset{\sim}{v}_B\Big|_{\text{rod}} = \underset{\sim}{v}_A + \overbrace{\underset{\sim}{\omega}\Big|_{\text{rod}}}^{-\dot{\psi}\hat{\underset{\sim}{e}}_z} \times \underset{\sim}{AB}$$

$$= R\dot{\varphi}\,\hat{\underset{\sim}{e}}_x + 3R\dot{\psi}\left(\sin\psi\,\hat{\underset{\sim}{e}}_x + \cos\psi\,\hat{\underset{\sim}{e}}_y\right)$$

$$= (R\dot{\varphi} + 3R\dot{\psi}\sin\psi)\,\hat{\underset{\sim}{e}}_x + 3R\dot{\psi}\cos\psi\,\hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{v}_B\Big|_{\text{left cyl.}} = \underset{\sim}{v}_B\Big|_{\text{rod}} \implies \begin{cases} (R + \varepsilon\cos\theta)\dot{\theta} = R\dot{\varphi} + 3R\dot{\psi}\,\overbrace{\sin\psi}^{\frac{\varepsilon\cos\theta}{3R}} \\ -\varepsilon\sin\theta\,\dot{\theta} = 3R\dot{\psi}\,\underbrace{\cos\psi}_{\frac{\sqrt{9R^2 - \varepsilon^2\cos^2\theta}}{3R}} \end{cases}$$

$$\implies \begin{cases} \dot{\psi} = \dfrac{-\varepsilon\sin\theta\,\dot{\theta}}{\sqrt{9R^2 - \varepsilon^2\cos^2\theta}} \\[2ex] \dot{\varphi} = \dfrac{(R + \varepsilon\cos\theta)\dot{\theta}}{R} + \dfrac{\varepsilon^2\sin\theta\cos\theta\,\dot{\theta}}{R\sqrt{9R^2 - \varepsilon^2\cos^2\theta}} \end{cases} \quad (*)$$

Problem 3

$$\underset{\sim}{v}_C = \underset{\sim}{v}_A + \underset{\sim}{\omega}\big|_{rod} \times \underset{\sim}{AC} = R\dot{\varphi}\, \hat{\underset{\sim}{e}}_x + \dot{\psi}\,\frac{3R}{2}\left(\sin\psi\, \hat{\underset{\sim}{e}}_x + \cos\psi\, \hat{\underset{\sim}{e}}_y\right)$$

$$= \left(R\dot{\varphi} + \frac{3R}{2}\dot{\psi}\sin\psi\right)\hat{\underset{\sim}{e}}_x + \frac{3R}{2}\dot{\psi}\cos\psi\, \hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{v}_C = \left[(R+\varepsilon\cos\theta)\,\dot{\theta} + \frac{\varepsilon^2 \sin\theta\cos\theta\,\dot{\theta}}{2\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right]\hat{\underset{\sim}{e}}_x - \frac{\varepsilon\sin\theta\,\dot{\theta}}{2}\,\hat{\underset{\sim}{e}}_y = G(\theta)\,\dot{\theta}\,\hat{\underset{\sim}{e}}_x - \frac{\varepsilon\sin\theta\,\dot{\theta}}{2}\,\hat{\underset{\sim}{e}}_y$$

There is one generalized coordinate $\xi_1 = \theta$ in this problem.

$$(*) \longrightarrow \qquad \delta\varphi = \left(\frac{R+\varepsilon\cos\theta}{R} + \frac{\varepsilon^2\sin\theta\cos\theta}{R\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right)\delta\theta$$

To find Ξ_θ, $\qquad \delta W_\theta = F(R\,\delta\varphi) = F\left(R+\varepsilon\cos\theta + \frac{\varepsilon^2\sin\theta\cos\theta}{\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right)\delta\theta$

$$\Longrightarrow \qquad \Xi_\theta = F\left(R+\varepsilon\cos\theta + \varepsilon^2\frac{\sin\theta\cos\theta}{\sqrt{9R^2-\varepsilon^2\cos^2\theta}}\right)$$

Construct Lagrangian: $\qquad \mathcal{L} = T - V$

$$T = \frac{1}{2}m\,\underset{\sim}{v}_O\cdot\underset{\sim}{v}_O + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{\theta}^2 + \frac{1}{2}m\,\underset{\sim}{v}_A\cdot\underset{\sim}{v}_A + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{\varphi}^2 + \frac{1}{2}M\,\underset{\sim}{v}_C\cdot\underset{\sim}{v}_C + \frac{1}{2}\left(\frac{1}{12}M(3R)^2\right)\dot{\psi}^2$$

$$T = \frac{1}{2}m(R\dot{\theta})^2 + \frac{1}{4}mR^2\dot{\theta}^2 + \frac{1}{2}m(R\dot{\varphi})^2 + \frac{1}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}M\left[G^2(\theta) + \frac{\varepsilon^2\sin^2\theta}{4}\right]\dot{\theta}^2 + \frac{3}{8}MR^2\frac{\varepsilon^2\sin^2\theta}{9R^2-\varepsilon^2\cos^2\theta}\dot{\theta}^2$$

$$T = \frac{3}{4}mR^2\dot{\theta}^2 + \frac{3}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}M\left[G^2(\theta) + \frac{\varepsilon^2\sin^2\theta}{4}\right]\dot{\theta}^2 + \frac{3}{8}MR^2\frac{\varepsilon^2\sin^2\theta}{9R^2-\varepsilon^2\cos^2\theta}\dot{\theta}^2$$

$$V = Mg\,y_C = Mg\left(\frac{3R}{2}\sin\psi\right) = Mg\,\frac{\varepsilon\cos\theta}{2} \qquad (y_O = 0,\ y_A = 0)$$

Problem 3

$$\therefore \quad \mathcal{L} = \frac{3}{4} m R^2 \dot{\theta}^2 + \frac{3}{4} m \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2 \dot{\theta}^2$$

$$+ \frac{1}{2} M \left\{ \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{2\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2 + \frac{\varepsilon^2 \sin^2\theta}{4} \right\} \dot{\theta}^2 + \frac{3}{8} M R^2 \frac{\varepsilon^2 \sin^2\theta}{9R^2 - \varepsilon^2 \cos^2\theta} \dot{\theta}^2$$

$$- Mg \frac{\varepsilon}{2} \cos\theta \quad = \quad H(\theta)\, \dot{\theta}^2 - Mg \frac{\varepsilon}{2} \cos\theta$$

$\delta\theta$:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \Xi_\theta$$

$$\frac{d}{dt}\left[2H(\theta)\, \dot{\theta} \right] - \left[\frac{dH}{d\theta} \dot{\theta}^2 + Mg \frac{\varepsilon}{2} \sin\theta \right] = \Xi_\theta$$

$$2 H(\theta)\, \ddot{\theta} + \frac{dH}{d\theta} \dot{\theta}^2 - Mg \frac{\varepsilon}{2} \sin\theta = \Xi_\theta$$

governing equation of motion

where

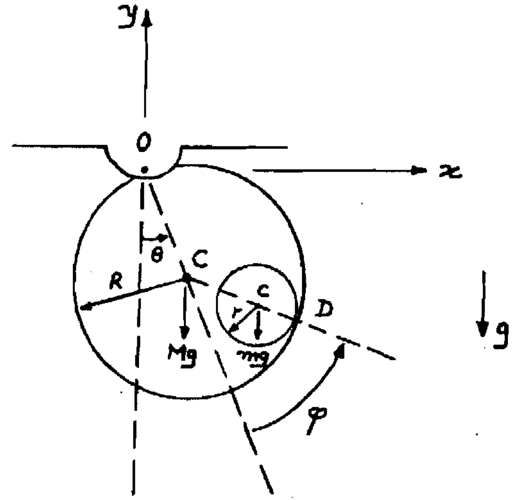
$$H(\theta) = \frac{3}{4} m R^2 + \frac{3}{4} m \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2$$

$$+ \frac{1}{2} M \left\{ \left[R + \varepsilon \cos\theta + \frac{\varepsilon^2 \sin\theta \cos\theta}{2\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right]^2 + \frac{\varepsilon^2 \sin^2\theta}{4} \right\} + \frac{3}{8} M R^2 \frac{\varepsilon^2 \sin^2\theta}{9R^2 - \varepsilon^2 \cos^2\theta}$$

and

$$\Xi_\theta = F \left(R + \varepsilon \cos\theta + \varepsilon^2 \frac{\sin\theta \cos\theta}{\sqrt{9R^2 - \varepsilon^2 \cos^2\theta}} \right)$$

Problem 4



Generalized coordinates:

$$\xi_1 = \theta \, , \qquad \xi_2 = \varphi$$

$$\underset{\sim}{\omega}_M = \underset{\sim}{\omega}_{ring} = \dot\theta \, \hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{\omega}_m = \underset{\sim}{\omega}_{disk} = \left[\dot\theta + \left(1 - \frac{R}{r}\right)\dot\varphi\right] \hat{\underset{\sim}{e}}_z$$

$$\underset{\sim}{v}_C = \underset{\sim}{v}_0 + \underset{\sim}{\omega}_{ring} \times \underset{\sim}{OC} = \underset{\sim}{0} + \dot\theta R \, \hat{\underset{\sim}{e}}_\theta = R\dot\theta \, \hat{\underset{\sim}{e}}_\theta$$

$$\begin{cases} x_c = R\sin\theta + (R-r)\sin(\theta+\varphi) \\ y_c = -R\cos\theta - (R-r)\cos(\theta+\varphi) \end{cases}$$

$$\Longrightarrow \quad \underset{\sim}{v}_c = \left[R\cos\theta \, \dot\theta + (R-r)\cos(\theta+\varphi)(\dot\theta+\dot\varphi)\right]\hat{\underset{\sim}{e}}_x + \left[R\sin\theta \, \dot\theta + (R-r)\sin(\theta+\varphi)(\dot\theta+\dot\varphi)\right]\hat{\underset{\sim}{e}}_y$$

Construct the Lagrangian:

$$\mathcal{L} = T - V$$

$$T = T_M + T_m$$

$$T_M = \frac{1}{2} M \underset{\sim}{v}_C \cdot \underset{\sim}{v}_C + \frac{1}{2} I_C \, \underset{\sim}{\omega}_M \cdot \underset{\sim}{\omega}_M = \frac{1}{2} M (R\dot\theta)^2 + \frac{1}{2}(MR^2)\dot\theta^2 = MR^2\dot\theta^2$$

$$T_m = \frac{1}{2} m \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c + \frac{1}{2} I_c \, \underset{\sim}{\omega}_m \cdot \underset{\sim}{\omega}_m = \frac{1}{2} m \left[R^2\dot\theta^2 + (R-r)^2(\dot\theta+\dot\varphi)^2 + 2R(R-r)\dot\theta(\dot\theta+\dot\varphi)\cos\varphi\right]$$
$$+ \frac{1}{2}\left(\frac{1}{2} m r^2\right)\left[\frac{r\dot\theta - (R-r)\dot\varphi}{r}\right]^2$$

$$V = Mg\,y_C + mg\,y_c = -MgR\cos\theta - mg\left[R\cos\theta + (R-r)\cos(\theta+\varphi)\right]$$
$$= -(M+m)gR\cos\theta - mg(R-r)\cos(\theta+\varphi)$$

Problem 4

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = 2MR^2\dot\theta + mR^2\dot\theta + m(R-r)^2(\dot\theta+\dot\varphi) + mR(R-r)\cos\varphi\,(2\dot\theta+\dot\varphi) + \frac{m}{2}r\left(r\dot\theta-(R-r)\dot\varphi\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -(M+m)gR\sin\theta - mg(R-r)\sin(\theta+\varphi)$$

$$\frac{\partial \mathcal{L}}{\partial \dot\varphi} = m(R-r)^2(\dot\theta+\dot\varphi) + mR\dot\theta(R-r)\cos\varphi - \frac{m}{2}(R-r)\left(r\dot\theta-(R-r)\dot\varphi\right)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -mR\dot\theta(\dot\theta+\dot\varphi)(R-r)\sin\varphi - mg(R-r)\sin(\theta+\varphi)$$

$\underline{\underline{\delta\theta}}$: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \Xi_\theta = 0$

$$\left[(2M+m)R^2 + m(R-r)^2 + 2mR(R-r)\cos\varphi + \frac{m}{2}r^2\right]\ddot\theta$$
$$+\left[m(R-r)\left(R-\frac{3r}{2}\right) + mR(R-r)\cos\varphi\right]\ddot\varphi - mR(R-r)\sin\varphi\left(2\dot\varphi\dot\theta+\dot\varphi^2\right)$$
$$+(M+m)gR\sin\theta + mg(R-r)\sin(\theta+\varphi) = 0$$

$\underline{\underline{\delta\varphi}}$: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\varphi}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = \Xi_\varphi = 0$

$$\left[m(R-r)\left(R-\frac{3}{2}r\right) + mR(R-r)\cos\varphi\right]\ddot\theta + \frac{3m}{2}(R-r)^2\ddot\varphi$$
$$+ mR(R-r)\sin\varphi\,\dot\theta^2 + mg(R-r)\sin(\theta+\varphi) = 0$$

Governing equations of motion