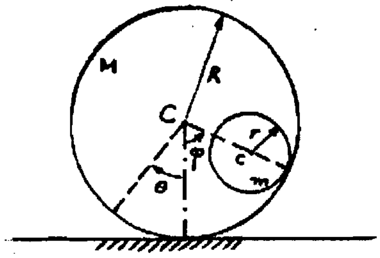
2.032 Dynamics

Problem Set No. 7

Problem 1



$\xi_1 = \theta$, $\xi_2 = \varphi$

$\underline{\underline{\delta\theta}}$:

$$\left(2M + \tfrac{3}{2}m\right)R^2\ddot{\theta} + mR(R-r)\left(\cos\varphi + \tfrac{1}{2}\right)\ddot{\varphi} - m\dot{\varphi}^2 R(R-r)\sin\varphi = 0$$

$\underline{\underline{\delta\varphi}}$:

$$\tfrac{3}{2}m(R-r)^2\ddot{\varphi} + mR(R-r)\ddot{\theta}\left(\cos\varphi + \tfrac{1}{2}\right) + mg(R-r)\sin\varphi = 0$$

(i) $\mathcal{L} = T - V$ depends on $\dot{\theta}$ but not θ and depends on both $\dot{\varphi}$ and φ.

$$\left.\begin{array}{l}\text{Here, } \varphi \text{ is non-ignorable} \\ \theta \text{ is ignorable}\end{array}\right\} \Longrightarrow \text{steady motion} \quad \begin{array}{l}\varphi = \varphi_s \text{ (Const.)} \\ \dot{\theta} = \Omega \text{ (Const.)}\end{array}$$

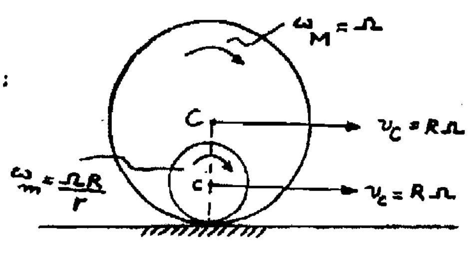
$$\delta\varphi \Longrightarrow mg(R-r)\sin\varphi_s = 0 \longrightarrow \varphi_s = 0, \pi$$

possible steady motions :

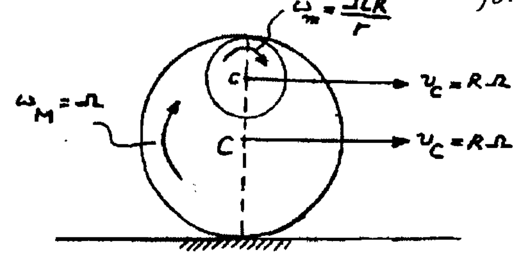
$$A: \quad \varphi_s = 0, \quad \dot{\theta} = \Omega$$
$$B: \quad \varphi_s = \pi, \quad \dot{\theta} = \Omega$$

Note that in finding the governing equations, we implicitly assumed that there is suitable "reaction" force between the cylinder and the tube to keep them attached. (Clearly, steady motion B is <u>unrealistic</u>; would require attractive reaction force)

(ii) $\underline{A}$:



$\underline{B}$:



Problem 1

(ii) In the steady motions A and B, center of mass of both cylinder and tube move with the same velocity ΩR although angular velocity of the cylinder is $\Omega R/r$. In the case A, lower points of both of them are instantaneously at rest. In the case B, upper points of both have the same velocity $2\Omega R$.

Note that rolling constraints are not violated and no slipping occurs.

(iii) Equilibrium position $\quad \varphi_s = 0, \quad \theta_s = 0 \qquad (\dot\theta = 0)$

$$\theta = \theta_s + \varepsilon_\theta(t) \approx \varepsilon_\theta(t) \qquad \varepsilon_\theta \ll 1 \qquad \dot\theta = \dot\varepsilon_\theta$$

$$\varphi = \varphi_s + \varepsilon_\varphi(t) = \varepsilon_\varphi(t) \qquad \varepsilon_\varphi \ll 1 \qquad \dot\varphi = \dot\varepsilon_\varphi$$

higher order term

$$\delta\theta \longrightarrow \quad \left(2M + \frac{3}{2}m\right) R^2 \ddot\varepsilon_\theta + mR(R-r)\left(1+\frac{1}{2}\right)\ddot\varepsilon_\varphi - mR(R-r)\,\dot\varepsilon_\varphi^2\,\varepsilon_\varphi = 0$$

$$\delta\varphi \longrightarrow \quad \frac{3}{2}m(R-r)^2\ddot\varepsilon_\varphi + mR(R-r)\left(1+\frac{1}{2}\right)\ddot\varepsilon_\theta + mg(R-r)\,\varepsilon_\varphi = 0$$

Eliminating $\ddot\varepsilon_\theta$,

$$\frac{3mM(R-r)^2}{2M + \frac{3}{2}m}\ddot\varepsilon_\varphi + mg(R-r)\,\varepsilon_\varphi = 0$$

$$\longrightarrow \quad \varepsilon_\varphi = A_\varphi \cos\omega_\varphi t + B_\varphi \sin\omega_\varphi t \qquad \text{where} \quad \omega_\varphi^2 = \left(\frac{2}{3} + \frac{1}{2}\frac{m}{M}\right)\frac{g}{R-r}$$

$$\begin{cases} \dot\varphi(0) = 0 \longrightarrow \dot\varepsilon_\varphi(t=0) = 0 \\ \varphi(0) = \varphi_0 \longrightarrow \varepsilon_\varphi(t=0) = 0 \end{cases} \Longrightarrow \begin{cases} B_\varphi = 0 \\ A_\varphi = \varphi_0 \end{cases} \Longrightarrow \quad \varepsilon_\varphi = \varphi_0 \cos\omega_\varphi t$$

$$\delta\theta \longrightarrow \quad \left(2M + \frac{3}{2}m\right)\ddot\varepsilon_\theta = -m\left(1 - \frac{r}{R}\right)\frac{3}{2}\ddot\varepsilon_\varphi = m\varphi_0\,\omega_\varphi^2\,\frac{3}{2}\left(1-\frac{r}{R}\right)\cos\omega_\varphi t$$

$$\longrightarrow \quad \ddot\varepsilon_\theta = \frac{mg}{2MR}\varphi_0 \cos\omega_\varphi t \quad \longrightarrow \quad \dot\varepsilon_\theta = \frac{mg}{2MR}\varphi_0\frac{1}{\omega_\varphi}\sin\omega_\varphi t + C_1$$

Problem 1

iii) $\dot{\varepsilon}_\theta(0) = 0 \longrightarrow C_1 = 0$

$$\Rightarrow \quad \varepsilon_\theta = -\frac{mg\varphi_0}{2MR\omega_\varphi^2}\cos\omega_\varphi t + C_2$$

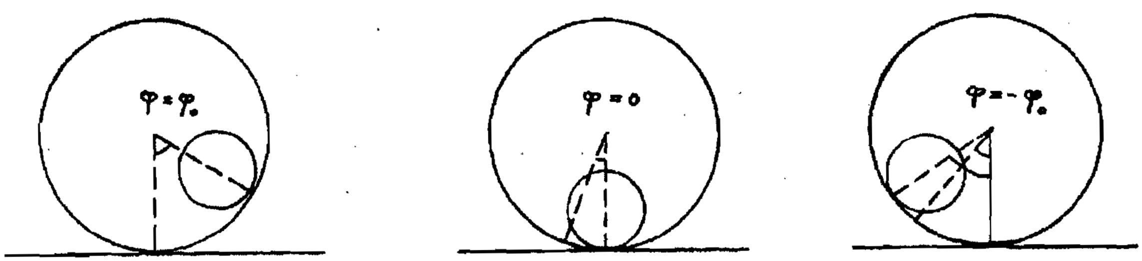
$$\varepsilon_\theta(0) = 0 \longrightarrow C_2 = \frac{mg\varphi_0}{2MR\omega_\varphi^2} \Rightarrow \boxed{\varepsilon_\theta = \varphi_0\frac{(1-r/R)}{(1+\frac{4}{3}M/m)}(1-\cos\omega_\varphi t)}$$

$\therefore \quad \varphi_s = \theta_s = 0$ is stable. Both ε_φ and ε_θ are oscillatory.

Both cylinder and tube move back and forth with the same frequency but with phase difference. When ε_φ is maximum (φ_0), ε_θ is minimum (0), and when ε_φ is minimum ($-\varphi_0$), ε_θ is maximum $2\varphi_0(1-\frac{r}{R})/(1+\frac{4}{3}\frac{M}{m})$.

Note that the maximum of ε_θ is less than the maximum of ε_φ (φ_0).

$$0 < \frac{r}{R}, \frac{m}{M} < 1 \longrightarrow 0 \leqslant \varepsilon_\theta < \frac{6}{7}\varphi_0$$



$\theta = 0$ $\qquad$ $\theta = \varphi_0\frac{(1-r/R)}{(1+\frac{4}{3}\frac{M}{m})}$ $\qquad$ $\theta = 2\varphi_0\frac{(1-r/R)}{(1+\frac{4}{3}\frac{M}{m})}$

iv) Full nonlinear equations can be written as:

$$\begin{cases} \left(2\frac{M}{m}+\frac{3}{2}\right)\ddot{\theta} + \left(1-\frac{r}{R}\right)\left(\frac{1}{2}+\cos\varphi\right)\ddot{\varphi} - \dot{\varphi}^2\left(1-\frac{r}{R}\right)\sin\varphi = 0 \\ \frac{3}{2}\left(1-\frac{r}{R}\right)^2\ddot{\varphi} + \left(1-\frac{r}{R}\right)\left(\frac{1}{2}+\cos\varphi\right)\ddot{\theta} + \frac{g}{R}\left(1-\frac{r}{R}\right)\sin\varphi = 0 \end{cases}$$

Problem 1

iv) Introducing new time $T = \sqrt{\frac{g}{R-r}}\, t$, we get

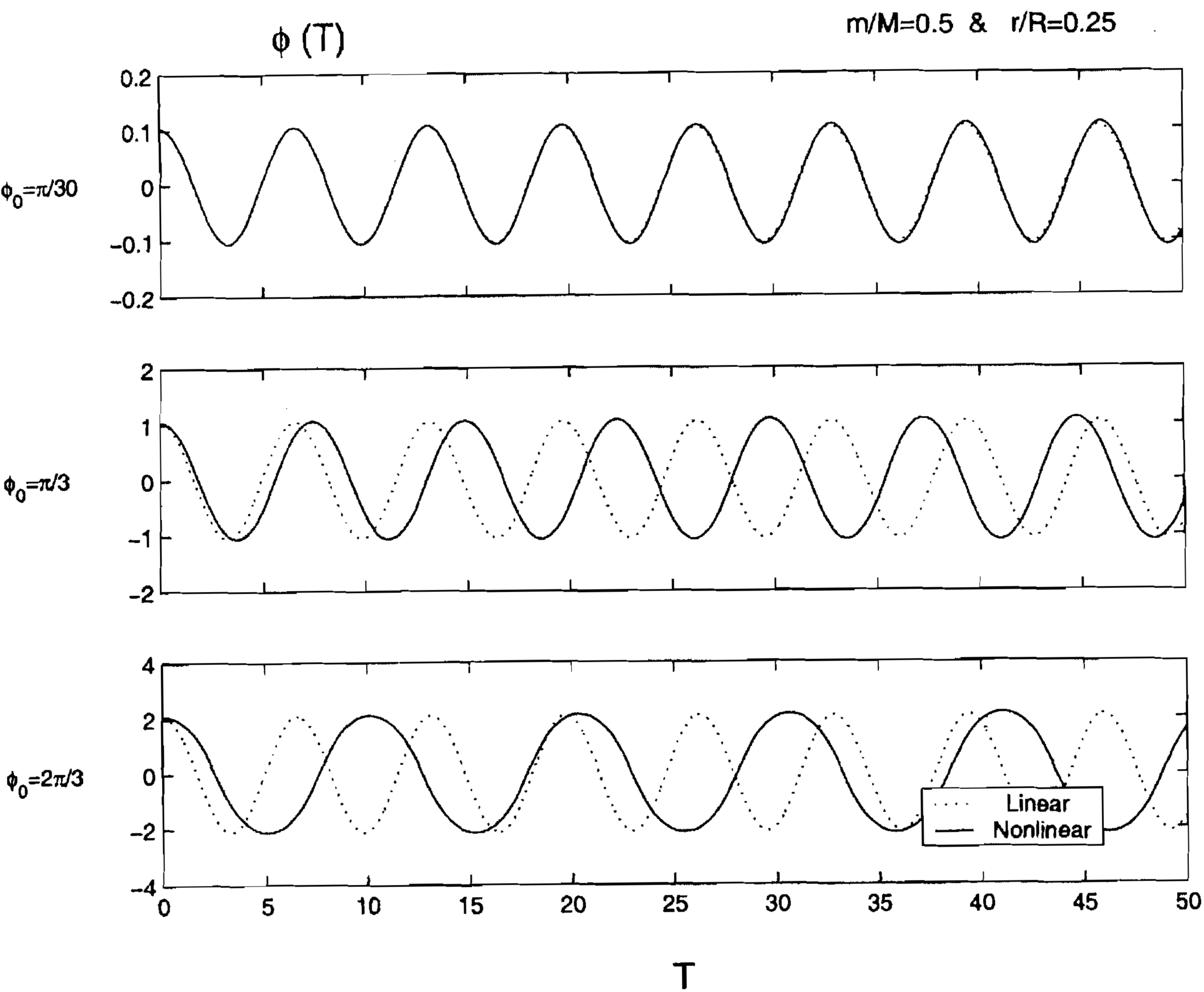
$$\begin{cases} \left(2\frac{M}{m} + \frac{3}{2}\right)\ddot{\theta} + \left(1 - \frac{r}{R}\right)\left(\frac{1}{2} + \cos\varphi\right)\ddot{\varphi} - \dot{\varphi}^2\left(1 - \frac{r}{R}\right)\sin\varphi = 0 \\ \frac{3}{2}\left(1 - \frac{r}{R}\right)^2\ddot{\varphi} + \left(1 - \frac{r}{R}\right)\left(\frac{1}{2} + \cos\varphi\right)\ddot{\theta} + \left(1 - \frac{r}{R}\right)^2\sin\varphi = 0 \end{cases}$$

$$\left(\ddot{\theta} = \frac{\partial^2\theta}{\partial T^2}, \quad \ddot{\varphi} = \frac{\partial^2\varphi}{\partial T^2}, \quad \dot{\varphi} = \frac{\partial\varphi}{\partial T}\right)$$

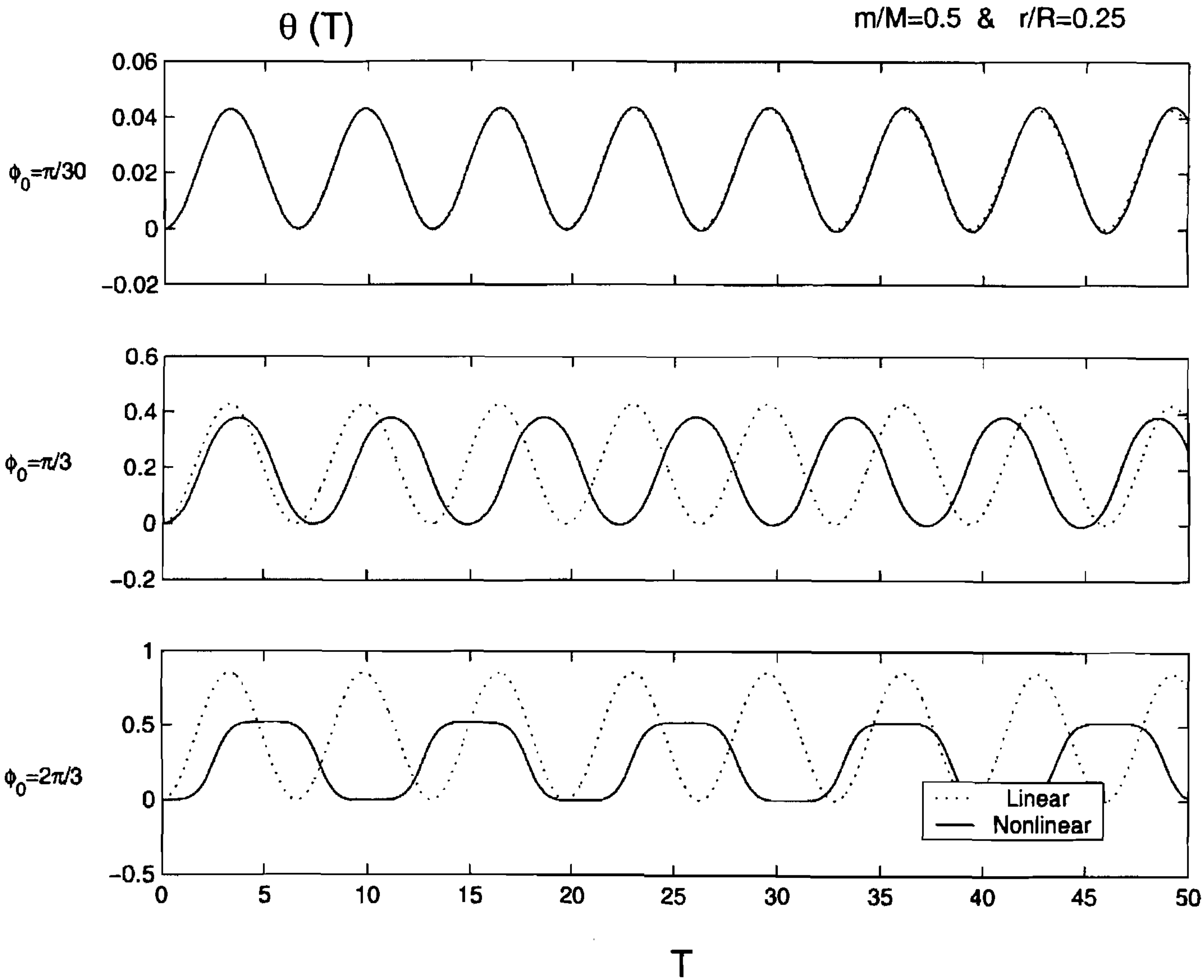
These Nonlinear equations are integrated numerically using forward finite difference method and the initial conditions in (iii). The results for $\frac{m}{M} = \frac{1}{2}$ and $\frac{r}{R} = \frac{1}{4}$ are shown in the following pages for three different values of φ_0.

It can be seen that for small φ_0, linear and nonlinear solutions agree but as φ_0 becomes large, they do not agree anymore; the larger the value of φ_0, the more the difference between the solutions. However, nonlinear solution is stable even for large values of φ_0 in this problem.

Problem 1


m/M=0.5 & r/R=0.25
ϕ (T)
ϕ0=π/30
ϕ0=π/3
ϕ0=2π/3
Linear
Nonlinear
T
0.2
0.1
0
-0.1
-0.2
2
1
0
-1
-2
4
2
0
-2
-4
0
5
10
15
20
25
30
35
40
45
50

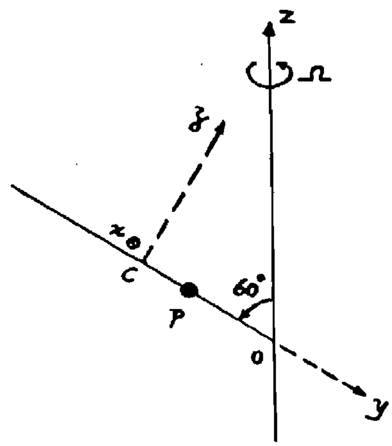
Problem 1


θ (T)
m/M=0.5 & r/R=0.25
$\phi_0=\pi/30$
$\phi_0=\pi/3$
$\phi_0=2\pi/3$
Linear
Nonlinear
T

Problem 2

$$\underset{\sim}{v}_p = \left[a\Omega\frac{\sqrt{3}}{2}(1-\cos\theta) + a\dot\theta\cos\theta\right]\hat{\underset{\sim}{e}}_x + \left[a\sin\theta\left(\Omega\frac{\sqrt{3}}{2}-\dot\theta\right)\right]\hat{\underset{\sim}{e}}_y + \left[a\frac{\Omega}{2}\sin\theta\right]\hat{\underset{\sim}{e}}_z$$

(Problem 2 of PS 3)

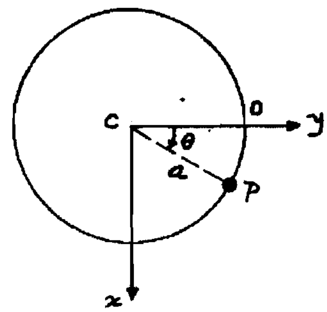


(i)

$$T = \frac{1}{2}m\underset{\sim}{v}_p\cdot\underset{\sim}{v}_p$$

$$= \frac{1}{2}ma^2\left\{\left[\Omega\frac{\sqrt{3}}{2}(1-\cos\theta) + \dot\theta\cos\theta\right]^2 + \left[\sin^2\theta\left(\Omega\frac{\sqrt{3}}{2}-\dot\theta\right)^2\right] + \frac{\Omega^2}{4}\sin^2\theta\right\}$$

$$V = -mg\,a\cos\theta\,\frac{1}{2}$$



$$\Xi_\theta = 0$$

$$\mathcal{L} = T - V$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = ma^2\left\{\frac{\sqrt{3}}{2}\Omega(\cos\theta-1) + \dot\theta\right\}$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = ma^2\left\{\frac{1}{4}\Omega^2\sin\theta(3+\cos\theta) - \frac{\sqrt{3}}{2}\Omega\dot\theta\sin\theta - \frac{g}{2a}\sin\theta\right\}$$

$\underline{\delta\theta}$:

$$\frac{d}{dt}\left\{\frac{\partial\mathcal{L}}{\partial\dot\theta}\right\} - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\ddot\theta - \frac{1}{4}\Omega^2\sin\theta(3+\cos\theta) + \frac{g}{2a}\sin\theta = 0$$

(ii)

θ is non-ignorable $\Longrightarrow$ steady motion $\theta = \theta_0$ (constant)

$$\delta\theta \Longrightarrow \sin\theta_0\left[\frac{g}{2a} - \frac{\Omega^2}{4}(3+\cos\theta_0)\right] = 0$$

Problem 2

(ii)

Possible equilibrium positions :

A: $\sin\theta_0 = 0$, $\cos\theta_0 = 1 \implies \theta_0 = 0$

B: $\sin\theta_0 = 0$, $\cos\theta_0 = -1 \implies \theta_0 = \pm\pi$

C: $\frac{\Omega^2}{2}(3+\cos\theta_0) = \frac{g}{a} \implies \cos\theta_0 = \frac{2g}{a\Omega^2} - 3 \longrightarrow \theta_0 = \pm\cos^{-1}\left(\frac{2g}{a\Omega^2}-3\right)$

possible only if $1 < \frac{g}{a\Omega^2} < 2$

$$\sqrt{\frac{g}{a}} > \Omega > \sqrt{\frac{g}{2a}}$$

Hold Ω constant and supply τ_z as needed.

$\theta = \theta_0 + \varepsilon(t)$

$$\delta\theta \implies \ddot{\varepsilon} - \frac{1}{4}\Omega^2(\sin\theta_0 + \varepsilon\cos\theta_0)(3+\cos\theta_0 - \varepsilon\sin\theta_0) + \frac{g}{2a}(\sin\theta_0 + \varepsilon\cos\theta_0) = 0$$

$O(1)$ terms cancel. $\left(\sin\theta_0\left[\frac{g}{2a} - \frac{\Omega^2}{4}(3+\cos\theta_0)\right] = 0 \right)$

$$\longrightarrow \quad \ddot{\varepsilon} - \frac{\Omega^2}{4}\cos\theta_0(3+\cos\theta_0)\varepsilon + \frac{\Omega^2}{4}\sin^2\theta_0\,\varepsilon + \frac{g}{2a}\cos\theta_0\,\varepsilon = 0$$

(A) $\ddot{\varepsilon} - \Omega^2\varepsilon + \frac{g}{2a}\varepsilon = 0$

$\ddot{\varepsilon} + \left(\frac{g}{2a} - \Omega^2\right)\varepsilon = 0$ unstable if $\frac{g}{2a} - \Omega^2 < 0 \longrightarrow \Omega > \sqrt{\frac{g}{2a}}$

(B) $\ddot{\varepsilon} + \frac{2}{4}\Omega^2\varepsilon - \frac{g}{2a}\varepsilon = 0$

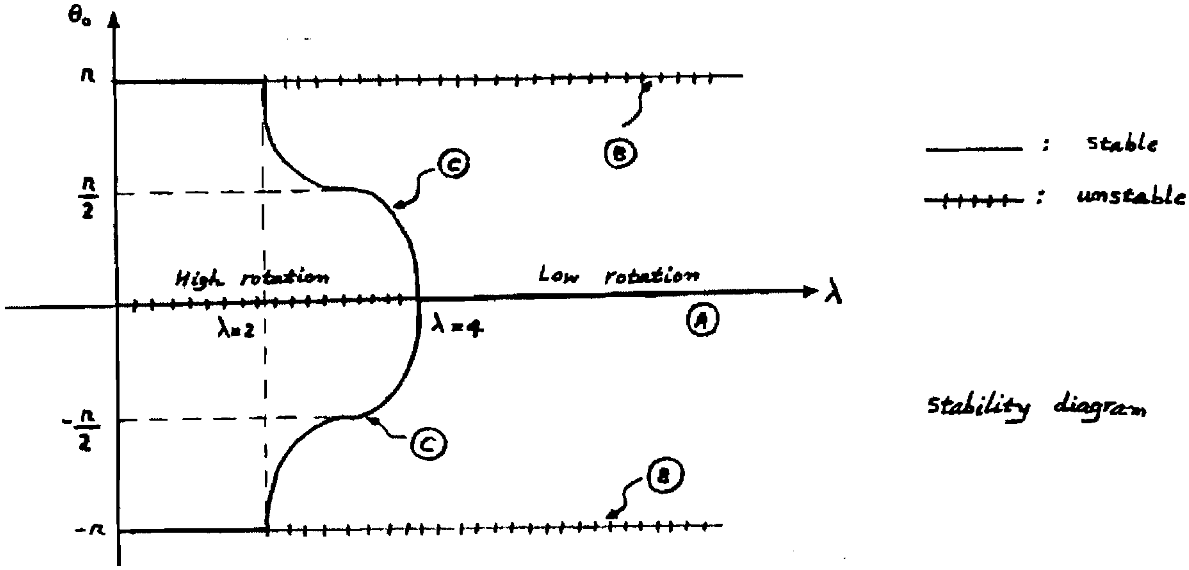
$\ddot{\varepsilon} + \left(\frac{\Omega^2}{2} - \frac{g}{2a}\right)\varepsilon = 0$ unstable if $\frac{\Omega^2}{2} - \frac{g}{2a} < 0 \longrightarrow \Omega < \sqrt{\frac{g}{a}}$

(C) $\ddot{\varepsilon} + \frac{\Omega^2}{4}\sin^2\theta_0\,\varepsilon = 0$ stable

Problem 2

iii) Define parameter $\lambda = \frac{2g}{a\Omega^2}$ $(0<\lambda<\infty)$

$\lambda_{crit} = 4, 2$ $\qquad \Omega_{crit} = \sqrt{\frac{g}{2a}}, \sqrt{\frac{g}{a}}$



Stability diagram

iv)

$$\ddot{\theta} - \frac{\Omega^2}{4}\sin\theta\,(3+\cos\theta) + \frac{g}{2a}\sin\theta = 0$$

Let $T = \sqrt{\frac{g}{a}}\,t$ $\longrightarrow$ $\ddot{\theta} - \frac{1}{2\lambda}\sin\theta\,(3+\cos\theta) + \frac{1}{2}\sin\theta = 0$

$$\longrightarrow \quad \ddot{\theta} + \frac{1}{2\lambda}(\lambda - 3 - \cos\theta)\sin\theta = 0$$

$$\begin{cases} \dfrac{d\theta}{dT} = y \\ \dfrac{dy}{dT} = \dfrac{1}{2\lambda}(\cos\theta + 3 - \lambda)\sin\theta \end{cases} \Longrightarrow \frac{dy}{d\theta} = \frac{1}{2\lambda y}(\cos\theta + 3 - \lambda)\sin\theta$$

$$\longrightarrow \quad y\,dy = \frac{1}{2\lambda}(\cos\theta + 3 - \lambda)\sin\theta\,d\theta$$

$$\longrightarrow \quad \frac{y^2}{2} = \frac{1}{2\lambda}\left[\frac{\sin^2\theta}{2} - (3-\lambda)\cos\theta\right] + C$$

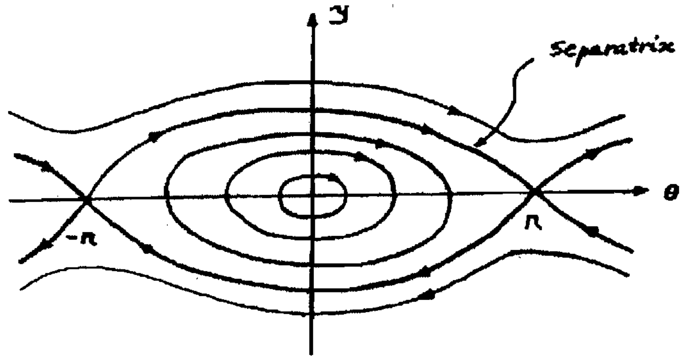
Problem 2

iv)

Low rotation

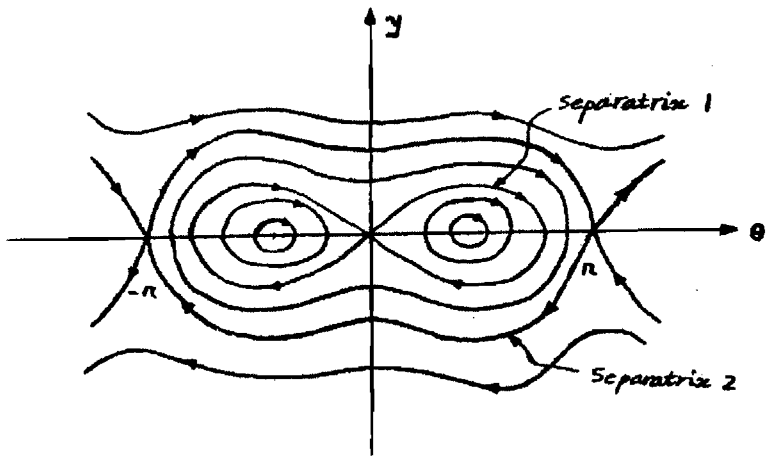
$\lambda > 4$

$\Omega < \sqrt{\frac{g}{2a}}$



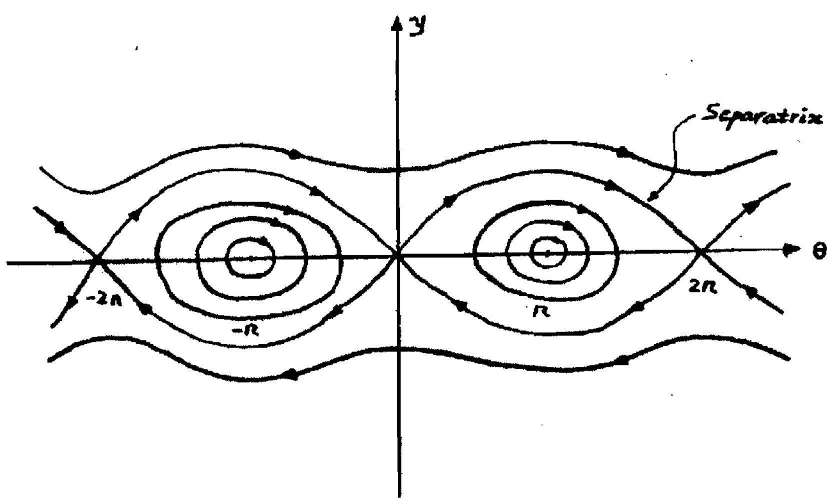
$4 > \lambda > 2$

$\sqrt{\frac{g}{a}} > \Omega > \sqrt{\frac{g}{2a}}$



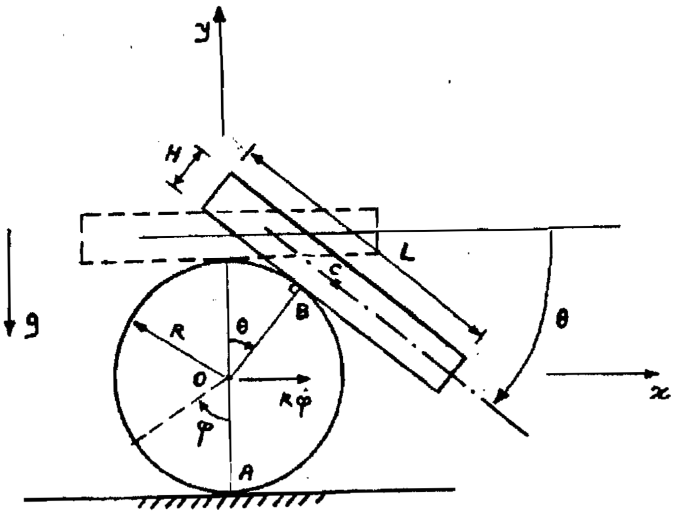
$0 < \lambda < 2$

$\Omega > \sqrt{\frac{g}{a}}$



v) If the ring is inclined at 120° to the vertical, just $\underline{\theta = 0}$ and $\underline{\theta = \pi}$ are equilibrium points, $\theta = 0$ being always <u>unstable</u> and $\theta = \pi$ being <u>stable</u> regardless of value of Ω. There are no equilibrium points between 0 and π in this case. The reason is that both gravity and rotation tend to move the bead toward $\theta = \pi$, whereas, in the 60° case, gravity tends to move the bead toward $\theta = 0$ while rotation tends to move it toward $\theta = \pi$.

Problem 3



) xy coordinate system moves with O.

$$\omega|_{cyl.} = -\dot{\varphi}\,\hat{\underset{\sim}{e}}_z$$

$$\omega|_{block} = -\dot{\theta}\,\hat{\underset{\sim}{e}}_z$$

No slip between the cylinder and the floor :

$$\underset{\sim}{v}_0 = R\dot{\varphi}\,\hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{v}_c = \underset{\sim}{v}_0 + \dot{x}_c\,\hat{\underset{\sim}{e}}_x + \dot{y}_c\,\hat{\underset{\sim}{e}}_y = (R\dot{\varphi} + \dot{x}_c)\,\hat{\underset{\sim}{e}}_x + \dot{y}_c\,\hat{\underset{\sim}{e}}_y$$

No slip between the block and the cylinder:

$$\underset{\sim}{v}_B|_{cyl.} = \underset{\sim}{v}_B|_{block}$$

$$\underset{\sim}{v}_B|_{cyl.} = \underset{\sim}{v}_0 + \omega|_{cyl.} \times \underset{\sim}{OB} = R\dot{\varphi}\,\hat{\underset{\sim}{e}}_x + (-\dot{\varphi}\,\hat{\underset{\sim}{e}}_z) \times (R\sin\theta\,\hat{\underset{\sim}{e}}_x + R\cos\theta\,\hat{\underset{\sim}{e}}_y)$$
$$= R\dot{\varphi}(1+\cos\theta)\,\hat{\underset{\sim}{e}}_x - R\dot{\varphi}\sin\theta\,\hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{v}_B|_{block} = \underset{\sim}{v}_c + \omega|_{block} \times \underset{\sim}{CB} = (R\dot{\varphi} + \dot{x}_c)\,\hat{\underset{\sim}{e}}_x + \dot{y}_c\,\hat{\underset{\sim}{e}}_y + (-\dot{\theta}\,\hat{\underset{\sim}{e}}_z) \times \big[(R\sin\theta - x_c)\,\hat{\underset{\sim}{e}}_x + (R\cos\theta - y_c)\,\hat{\underset{\sim}{e}}_y\big]$$

$$= (R\dot{\varphi} + \dot{x}_c + R\dot{\theta}\cos\theta - \dot{\theta}y_c)\,\hat{\underset{\sim}{e}}_x + (\dot{y}_c - R\dot{\theta}\sin\theta + \dot{\theta}x_c)\,\hat{\underset{\sim}{e}}_y$$

$$\therefore \begin{cases} R\dot{\varphi}\cos\theta = \dot{x}_c + R\dot{\theta}\cos\theta - \dot{\theta}y_c \\ -R\dot{\varphi}\sin\theta = \dot{y}_c - R\dot{\theta}\sin\theta + \dot{\theta}x_c \end{cases} \Longrightarrow \begin{cases} \sin\theta\,\dot{x}_c + \cos\theta\,\dot{y}_c - \dot{\theta}y_c\sin\theta + \dot{\theta}x_c\cos\theta = 0 \\ R(\dot{\varphi} - \dot{\theta}) = \cos\theta\,\dot{x}_c - \sin\theta\,\dot{y}_c - \dot{\theta}\cos\theta\,y_c - \dot{\theta}x_c\sin\theta \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{d}{dt}(\sin\theta\,x_c + \cos\theta\,y_c) = 0 \\ \dfrac{d}{dt}[R(\varphi - \theta)] = \dfrac{d}{dt}[\cos\theta\,x_c - \sin\theta\,y_c] \end{cases}$$

Problem 3

(a)

$$\Longrightarrow \begin{cases} \sin\theta\, x_c + \cos\theta\, y_c = \text{Const.} = R + \frac{H}{2} \\ \cos\theta\, x_c - \sin\theta\, y_c = R(\varphi - \theta) \end{cases} \qquad \left(\text{at } \theta = 0,\ x_c = 0,\ y_c = R + \frac{H}{2}\right)$$

$$\longrightarrow \begin{cases} x_c = R(\varphi-\theta)\cos\theta + \left(R+\frac{H}{2}\right)\sin\theta \\ y_c = \left(R+\frac{H}{2}\right)\cos\theta - R(\varphi-\theta)\sin\theta \end{cases}$$

(b)

$\xi_1 = \varphi$ & $\xi_2 = \theta$ complete set of independent generalized coordinates

Mass of the block $= m$ $\qquad I_{block} = \frac{1}{12} m (H^2 + L^2)$

$$T = \frac{1}{2} M\, \underset{\sim}{v}_0 \cdot \underset{\sim}{v}_0 + \frac{1}{2} I_{cyl.}\, \underset{\sim}{\omega}_{cyl.} \cdot \underset{\sim}{\omega}_{cyl.} + \frac{1}{2} m\, \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c + \frac{1}{2} I_{block}\, \underset{\sim}{\omega}_{block} \cdot \underset{\sim}{\omega}_{block}$$

$$T = \frac{1}{2} M (R\dot{\varphi})^2 + \frac{1}{2}\left(\frac{1}{2} M R^2\right)\dot{\varphi}^2 + \frac{1}{2} m \left[(R\dot{\varphi} + \dot{x}_c)^2 + \dot{y}_c^2\right] + \frac{1}{2}\left[\frac{1}{12} m (H^2 + L^2)\right]\dot{\theta}^2$$

$$V = m g y_c + M g \cancel{y_0}^{\,0} = m g y_c$$

$$T = \frac{3}{4} M R^2 \dot{\varphi}^2 + \frac{1}{2} m \Big[R^2\dot{\varphi}^2 + R^2(\varphi-\theta)^2\dot{\theta}^2 + R^2(\dot{\varphi}-\dot{\theta})^2 + \left(R+\frac{H}{2}\right)^2\dot{\theta}^2 - 2R^2\dot{\theta}\dot{\varphi}(\varphi-\theta)\sin\theta$$
$$+ 2R^2\dot{\varphi}^2\cos\theta + RH\dot{\varphi}\dot{\theta}\cos\theta + 2R(\dot{\varphi}-\dot{\theta})\left(R+\frac{H}{2}\right)\dot{\theta} \Big] + \frac{1}{24} m (H^2 + L^2)\dot{\theta}^2$$

$$V = mg\left[\left(R+\frac{H}{2}\right)\cos\theta - R(\varphi-\theta)\sin\theta\right]$$

$$\mathcal{L} = T - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{3}{2} M R^2 \dot{\varphi} + \frac{m}{2}\left[2R^2\dot{\varphi} + 2R^2(\dot{\varphi}-\dot{\theta}) - 2R^2(\varphi-\theta)\dot{\theta}\sin\theta + 2R^2(2\dot{\varphi})\cos\theta + RH\dot{\theta}\cos\theta + 2R\left(R+\frac{H}{2}\right)\dot{\theta}\right]$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{m}{2}\left[2R^2\dot{\theta}^2(\varphi-\theta) - 2R^2\dot{\theta}\dot{\varphi}\sin\theta\right] + mgR\sin\theta$$

Problem 3

(b)

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{m}{2}\left[2R^2(\varphi-\theta)^2\dot\theta - 2R^2(\dot\varphi-\dot\theta) + 2\left(R+\frac{H}{2}\right)^2\dot\theta^2 - 2R^2\dot\varphi(\varphi-\theta)\sin\theta + RH\dot\varphi\cos\theta\right.$$
$$\left. + 2R(\dot\varphi-2\dot\theta)\left(R+\frac{H}{2}\right)\right] + \frac{m}{12}(H^2+L^2)\dot\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{m}{2}\left[2R^2\dot\theta^2(\theta-\varphi) - 2R^2\dot\theta\dot\varphi\,\varphi\cos\theta + 2R^2\dot\theta\dot\varphi\sin\theta + 2R^2\dot\theta\dot\varphi\,\theta\cos\theta - 2R^2\dot\varphi^2\sin\theta\right.$$
$$\left. - RH\dot\varphi\dot\theta\sin\theta\right] + mg\left(R+\frac{H}{2}\right)\sin\theta + mgR\varphi\cos\theta - mgR\theta\cos\theta - mgR\sin\theta$$

$$\Xi_\varphi = \Xi_\theta = 0$$

$\underline{\underline{\delta\varphi}}$:

$$\left[\frac{3}{2}MR^2 + 2mR^2(1+\cos\theta)\right]\ddot\varphi + \left[-2mR^2\varphi\sin\theta + \frac{mRH}{2}(\cos\theta+1)\right]\ddot\theta + \cdots - mgR\sin\theta = 0$$

$\underline{\underline{\delta\theta}}$:

$$\left[mR^2(\varphi-\theta)^2 + m\left(\frac{H^2}{3}+\frac{L^2}{12}\right)\right]\ddot\theta + \left[-mR^2(\varphi-\theta)\sin\theta + \frac{mRH}{2}(\cos\theta+1)\right]\ddot\varphi + \cdots - mg\frac{H}{2}\sin\theta + mgR\cos\theta(\theta-\varphi) = 0$$

Governing Equations of Motion

Note that missing terms do not play a role in the following linear stability analysis

(c)

θ and φ are non-ignorable coordinates. $\Longrightarrow$ equilibrium position $\theta=\theta_s$, $\varphi=\varphi_s$

$$\delta\varphi \Longrightarrow -mgR\sin\theta_s = 0 \longrightarrow \theta_s = 0 \qquad \cancel{\theta_s=\pi}$$

$$\delta\theta \Longrightarrow -mg\frac{H}{2}\sin\theta_s + mgR\cos\theta_s(\theta_s-\varphi_s) = 0 \xrightarrow{\theta_s=0} \varphi_s = 0$$

$\therefore$ Equilibrium position $\underline{\theta_s = \varphi_s = 0}$

Problem 3

(c)

$$\theta = \theta_S + \varepsilon_\theta(t) = \varepsilon_\theta(t)$$

$$\varphi = \varphi_S + \varepsilon_\varphi(t) = \varepsilon_\varphi(t)$$

$$\delta\theta \implies \left(4mR^2 + \frac{3}{2}MR^2\right)\ddot{\varepsilon}_\varphi + mRH\,\ddot{\varepsilon}_\theta - mgR\,\varepsilon_\theta = 0$$

$$\delta\varphi \implies \left(m\frac{H^2}{3} + m\frac{L^2}{12}\right)\ddot{\varepsilon}_\theta + mRH\,\ddot{\varepsilon}_\varphi + mg\left(R - \frac{H}{2}\right)\varepsilon_\theta - mgR\,\varepsilon_\varphi = 0$$

$$\begin{bmatrix} R^2\left(\frac{3}{2}M + 4m\right) & mRH \\ mRH & m\left(\frac{H^2}{3} + \frac{L^2}{12}\right) \end{bmatrix} \begin{Bmatrix} \ddot{\varepsilon}_\varphi \\ \ddot{\varepsilon}_\theta \end{Bmatrix} + \begin{bmatrix} 0 & -mgR \\ -mgR & mg\left(R - \frac{H}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_\varphi \\ \varepsilon_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

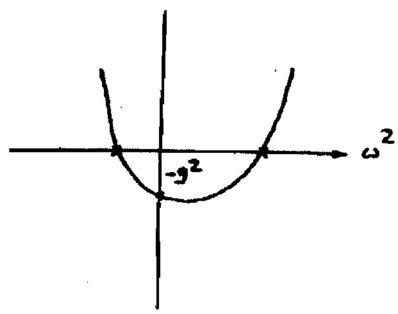
Let $\begin{Bmatrix} \varepsilon_\varphi \\ \varepsilon_\theta \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} e^{i\omega t}$

$$\therefore \begin{bmatrix} -\omega^2 R^2\left(\frac{3}{2}M + 4m\right) & -mR\left(g + \omega^2 H\right) \\ -mR\left(g + \omega^2 H\right) & +mg\left(R - \frac{H}{2}\right) - \omega^2 m\left(\frac{H^2}{3} + \frac{L^2}{12}\right) \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Non-trivial solution : $\begin{vmatrix} -\omega^2 R^2\left(\frac{3}{2}M + 4m\right) & -mR\left(g + \omega^2 H\right) \\ -mR\left(g + \omega^2 H\right) & mg\left(R + \frac{H}{2}\right) - \omega^2 m\left(\frac{H^2}{3} + \frac{L^2}{12}\right) \end{vmatrix} = 0$

$$\implies \left[\frac{M}{m}\left(\frac{H^2}{2} + \frac{L^2}{8}\right) + \frac{1}{3}\left(H^2 + L^2\right)\right]\omega^4 + \left[-4Rg + \frac{3}{4}\frac{M}{m}Hg - \frac{3}{2}\frac{M}{m}Rg\right]\omega^2 - g^2 = 0$$

This equation has a negative root for ω^2.



so $\varphi_S = \theta_S = 0$ is <u>unstable</u>.

Problem 3

Note that in the limiting case of $\frac{M}{m} \to \infty$, the problem turns into the rolling block on a fixed cylinder. In that case, we get

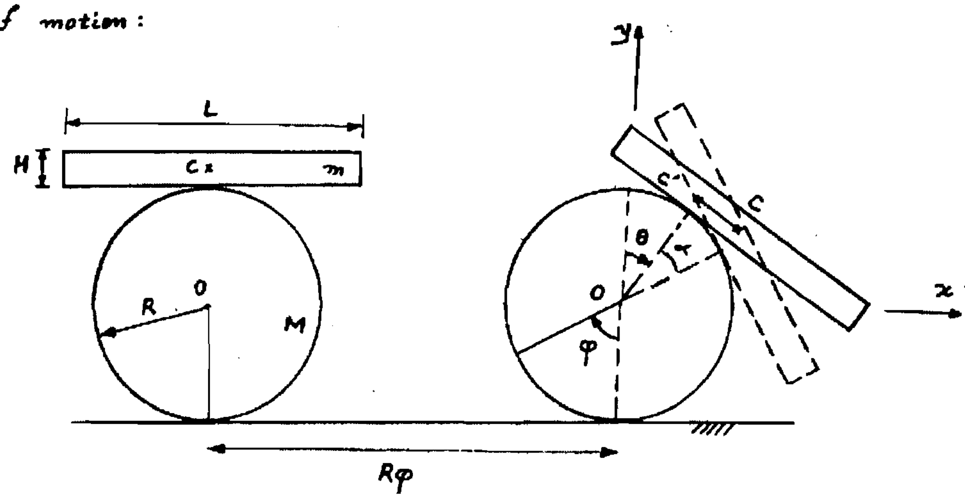
$$\left[\left(\frac{H^2}{2}+\frac{L^2}{8}\right)+\cancel{\frac{1}{3}\frac{m}{M}\left(H^2+L^2\right)}^{\,0}\right]\omega^4+\left[\cancel{-4Rg\frac{m}{M}}^{\,0}+\frac{3}{4}Hg-\frac{3}{2}Rg\right]\omega^2-\cancel{\frac{m}{M}g^2}^{\,0}=0$$

$$\left(\frac{H^2}{2}+\frac{L^2}{8}\right)\omega^2+\frac{3}{4}g\left(H-2R\right)=0$$

$$\longrightarrow \quad \omega^2=\frac{\frac{3}{4}g\,(2R-H)}{\frac{H^2}{2}+\frac{L^2}{8}} \quad \longrightarrow \quad \begin{cases} \theta_s=0 \text{ is stable for } \underline{H<2R} \\ \theta_s=0 \text{ is unstable for } \underline{H>2R} \end{cases}$$

* Another way to find the location of the center of mass c of the block (x_c, y_c) by studying the geometry of motion:

To analyze the block rolling constraint, assume the block is glued to the cylinder during rotation φ and then rolls back to location θ.



Let roll-back angle be α $(\alpha=\varphi-\theta)$. The absolute rotation of the block is through angle θ.

Location of center of mass of the block: (No slip between the block and the cylinder yields $cc' = R\alpha = R(\varphi-\theta)$)

$$\begin{cases} x_c = \left(R+\frac{H}{2}\right)\sin\theta + R(\varphi-\theta)\cos\theta \\ y_c = \left(R+\frac{H}{2}\right)\cos\theta - R(\varphi-\theta)\sin\theta \end{cases}$$