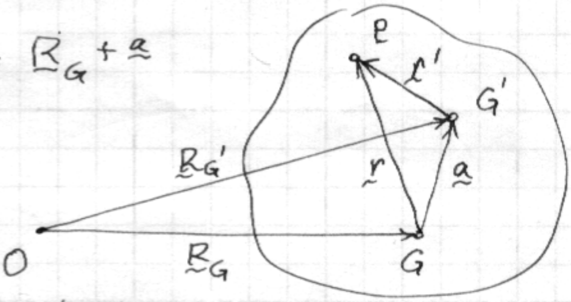
STAEDTLER® No. 937 811E Engineer's Computation Pad

Remarks

* Angular velocity $\boldsymbol{\omega}$ is independent of the choice of ref. pt G

$$\mathbf{r} = \mathbf{a} + \mathbf{r}', \quad \mathbf{R}_{G'} = \mathbf{R}_G + \mathbf{a}$$



For any pt P on the body,

$$\left.\begin{aligned} \mathbf{v}_P &= \mathbf{v}_G + \boldsymbol{\omega}\times\mathbf{r} \\ \mathbf{v}_P &= \mathbf{v}_{G'} + \boldsymbol{\omega}'\times\mathbf{r}' \end{aligned}\right\} \quad \text{But } \mathbf{v}_{G'} = \mathbf{v}_G + \boldsymbol{\omega}\times\mathbf{a}$$

$$\therefore \quad \boldsymbol{\omega}\times\mathbf{r} = \boldsymbol{\omega}\times\underbrace{\mathbf{a}}_{\mathbf{r}-\mathbf{r}'} + \boldsymbol{\omega}'\times\mathbf{r}'$$

$$\Rightarrow \quad (\boldsymbol{\omega}-\boldsymbol{\omega}')\times\mathbf{r}' = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{\omega} = \boldsymbol{\omega}'} \quad (\text{since } \mathbf{r}' \text{ is arbitrary})$$

* Since $\mathbf{v}_{G'} = \mathbf{v}_G + \boldsymbol{\omega}\times\mathbf{a}$ and $\boldsymbol{\omega}\times\mathbf{a} \perp \boldsymbol{\omega}$,

it is possible to choose G' s.t. $\mathbf{v}_{G'} \parallel \boldsymbol{\omega}$.

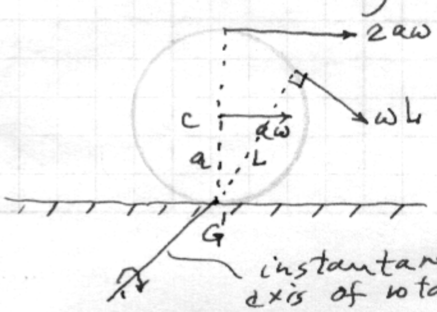
Then,

$$\mathbf{v}_P = \underbrace{\mathbf{v}_{G'}}_{\substack{\text{translation}\\ \text{along axis}\\ \text{of rotation}}} + \underbrace{\boldsymbol{\omega}\times\mathbf{r}'}_{\substack{\text{pure rotation about}\\ \text{axis passing through } G'}}$$

– If $\mathbf{v}_{G'} = 0$, the motion is (instantaneously) pure rotation about axis through G'

– In plane motion, $\mathbf{v}_{G'} = 0$, so instantaneous axis of pure rotation can always be found

e.g. rolling disk (no slip)



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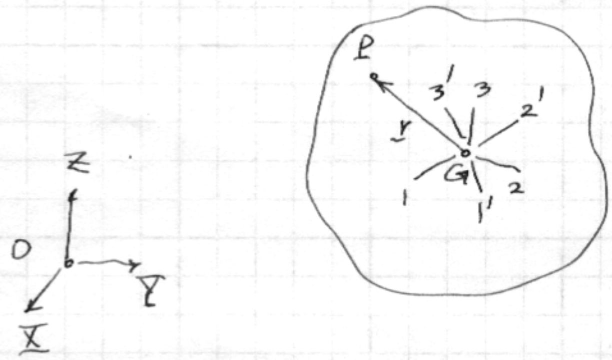
* Angular velocity $\underset{\sim}{\omega}$ is independent of choice of body-fixed frame G123



- Suppose we had chosen G1'2'3' instead, where

$$\{x_1\} = [R]\{x_1'\} \; ; \; [R] \text{ is orthogonal: } [R]^{-1} = [R]^{tr}$$

and
$$\{X_P\} = \{X_G\} + [T']\{x_1'\} = \{X_G\} + \{\tilde{x}_1\}$$

- Then,
$$\frac{d}{dt}\{X_P\} = \frac{d}{dt}\{X_G\} + \underbrace{\left[\frac{dT'}{dt}\right]\{x_1'\}}_{\displaystyle = \left(\left[\frac{dT'}{dt}\right][T']^{tr}\right)\{\tilde{x}_1\} = \underset{\sim}{\omega}' \times \underline{r}}$$

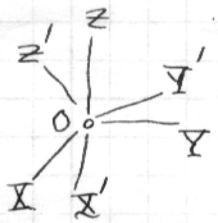
- But
$$\{\tilde{x}_1\} = [T']\{x_1'\} = [T]\{x_1\} = \left([T][R]\right)\{x_1'\}$$

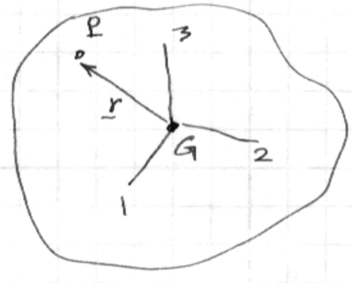
ie,
$$[T'] = [T][R]$$

- Hence,
$$\left[\frac{dT'}{dt}\right][T']^{tr} = \left[\frac{dT}{dt}\right]\underbrace{\cancel{[R][R]^{tr}}}_{[1]}[T]^{tr}$$
$$= \left[\frac{dT}{dt}\right][T]^{tr} \Rightarrow \boxed{\underset{\sim}{\omega} = \underset{\sim}{\omega}'}$$

$$\therefore \quad \underset{\sim}{\omega} \times \underline{r} = \underset{\sim}{\omega}' \times \underline{r} \quad \text{(as expected)}$$

* Angular velocity ω transforms like a vector upon rotations of inertial frame $OXYZ$





– Suppose we had chosen $OX'Y'Z'$ instead, where

$$\{X'_P\} = [R]\{X_P\}, \quad \{X'_G\} = [R]\{X_G\}$$
$$[R]^{tr} = [R]^{-1}$$

– Then,

$$\{X_P\} = \{X_G\} + [T]\{x_1\}$$
$$\{X'_P\} = \{X'_G\} + [T']\{x_1\}$$

– But $\quad \{\tilde{x}_1'\} = [T']\{x_1\}$ and $\{\tilde{x}_1\} = [T]\{x_1\}$

are related via $\quad \{\tilde{x}_1'\} = [R]\{\tilde{x}_1\} \Rightarrow$

$$[T']\{x_1\} = [R][T]\{x_1\} \Rightarrow \underline{\underline{[T'] = [R][T]}}$$

– Now,

$$\underbrace{\frac{d}{dt}\{X'_P\}}_{=\,[R]\frac{d}{dt}\{X_P\}} = \underbrace{\frac{d}{dt}\{X'_G\}}_{=\,[R]\frac{d}{dt}\{X_G\}} + \left(\left[\frac{dT'}{dt}\right][T']^{tr}\right)\underbrace{\{\tilde{x}_1'\}}_{=\,[R]\{\tilde{x}_1\}}$$

– How does $\left[\frac{dT'}{dt}\right][T']^{tr}$ transform ?

$$\left[\frac{dT'}{dt}\right][T']^{tr} = [R]\left(\left[\frac{dT}{dt}\right][T]^{tr}\right)[R]^{tr} \quad \text{ie,}$$

$$[\Omega] \equiv \left[\frac{dT}{dt}\right][T]^{tr} \quad \text{transforms like a } \underline{\text{tensor}}$$

But, since $[\Omega]$ is antisymmetric, it can be shown that also $\{\omega\}$ transforms like a vector : $\quad \underline{\underline{\{\omega'\} = [R]\{\omega\}}}$