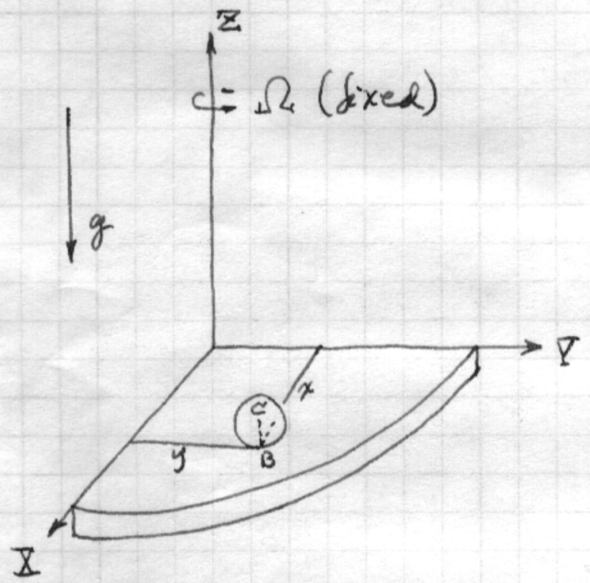
No. 937 811E Engineer's Computation Pad STAEDTLER

4.1 Motion of a ball rolling on a turntable



Ball has mass m, radius a

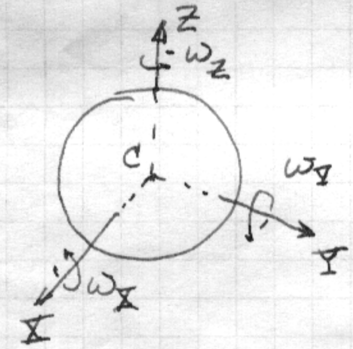
$$I_C = \frac{2}{5} m a^2$$

XYZ is inertial frame

Assume pure rolling, no slip

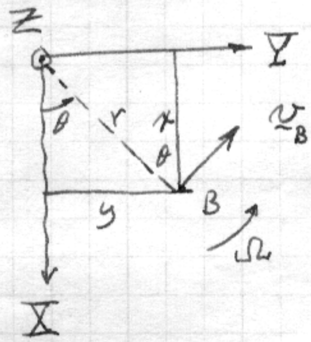
* Geometry motion — Center of ball C has coordinates (x, y, a)

Ball has angular velocity $\underset{\sim}{\omega} = \omega_X \hat{\underset{\sim}{e}}_X + \omega_Y \hat{\underset{\sim}{e}}_Y + \omega_Z \hat{\underset{\sim}{e}}_Z$



$$\underset{\sim}{v}_C = \dot{x}\, \hat{\underset{\sim}{e}}_X + \dot{y}\, \hat{\underset{\sim}{e}}_Y$$

* Impose pure-rolling constraint: $\underset{\sim}{v}_B \big|_{table} = \underset{\sim}{v}_B \big|_{ball}$



$$\underset{\sim}{v}_B \big|_{table} = \Omega r \hat{\underset{\sim}{e}}_\theta = -y\Omega \hat{\underset{\sim}{e}}_X + x\Omega \hat{\underset{\sim}{e}}_Y$$

Also, $\boldsymbol{v}_B\big|_{ball} = \boldsymbol{v}_C + \boldsymbol{\omega}\times\boldsymbol{a}$ with $\boldsymbol{a} = -a\,\hat{\boldsymbol{e}}_Z$

$$\boldsymbol{\omega}\times\boldsymbol{a} = a\,\omega_X\,\hat{\boldsymbol{e}}_Y - a\,\omega_Y\,\hat{\boldsymbol{e}}_X$$

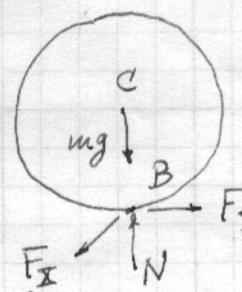
$$\therefore\quad \boldsymbol{v}_B\big|_{ball} = (\dot{x} - a\,\omega_Y)\,\hat{\boldsymbol{e}}_X + (\dot{y} + a\,\omega_X)\,\hat{\boldsymbol{e}}_Y$$

pure rolling then $\Rightarrow \begin{cases} -y\Omega = \dot{x} - a\,\omega_Y \Rightarrow \underline{\underline{\omega_Y = \dfrac{\Omega y}{a} + \dfrac{\dot{x}}{a}}} \\ x\Omega = \dot{y} + a\,\omega_X \Rightarrow \underline{\underline{\omega_X = \dfrac{\Omega x}{a} - \dfrac{\dot{y}}{a}}} \end{cases}$

* <u>Apply angular momentum principle about contact pt B</u>

(<u>moving with</u> $\boldsymbol{v}_B\big|_{contact}$)

$$\boldsymbol{\tau}_B = \frac{d}{dt}\boldsymbol{H}_B + \boldsymbol{v}_B\big|_{contact}\times\boldsymbol{P}$$



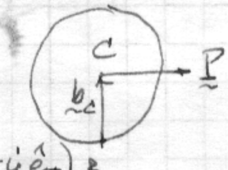
<u>Note</u>: $\boldsymbol{v}_B\big|_{contact} \neq \boldsymbol{v}_B\big|_{ball} = \boldsymbol{v}_B\big|_{table}$

contact pt B: $(x, y, 0) \Rightarrow \boldsymbol{v}_B\big|_{contact} = \boldsymbol{v}_C = \dot{x}\,\hat{\boldsymbol{e}}_X + \dot{y}\,\hat{\boldsymbol{e}}_Y$

($B\big|_{contact}$ is <u>not</u> a material pt on ball or table)

Clearly, $\boldsymbol{v}_B\big|_{cont} \parallel \boldsymbol{P} = m\,\boldsymbol{v}_C \Rightarrow \boldsymbol{v}_B\big|_{contact}\times\boldsymbol{P} = 0$

Also $\boldsymbol{\tau}_B = \boldsymbol{0} \Rightarrow \underline{\underline{\boldsymbol{H}_B = \text{constant}}}$



Now, $\boldsymbol{H}_B = \boldsymbol{H}_C + \boldsymbol{b}_c\times\boldsymbol{P}$, $\boldsymbol{H}_C = \boldsymbol{I}_C\,\boldsymbol{\omega}$

$$\boldsymbol{b}_C\times\boldsymbol{P} = a\,\hat{\boldsymbol{e}}_Z\times m\,(\dot{x}\,\hat{\boldsymbol{e}}_X + \dot{y}\,\hat{\boldsymbol{e}}_Y) = am\,(\dot{x}\,\hat{\boldsymbol{e}}_Y - \dot{y}\,\hat{\boldsymbol{e}}_X)$$

Therefore $\boldsymbol{H}_B = \text{const} \Rightarrow \begin{cases} I_C\,\omega_Z = \text{const} & ① \\ I_C\,\omega_X - am\dot{y} = \text{const} & ② \\ I_C\,\omega_Y + am\dot{x} = \text{const} & ③ \end{cases}$

Using $\omega_X = \frac{\Omega x}{a} - \frac{\dot{y}}{a}$, $\omega_Y = \frac{\Omega y}{a} + \frac{\dot{x}}{a}$,

② $\Rightarrow$ $\frac{2}{5} ma^2 \left(\frac{\Omega x}{a} - \frac{\dot{y}}{a}\right) - am\dot{y} = \text{const} \Rightarrow \underline{\underline{\frac{7}{5}\dot{y} - \frac{2}{5}x\Omega + C_1 = 0}}$

③ $\Rightarrow$ $\frac{2}{5} ma^2 \left(\frac{\Omega y}{a} + \frac{\dot{x}}{a}\right) + am\dot{x} = \text{const} \Rightarrow \underline{\underline{\frac{7}{5}\dot{x} + \frac{2}{5}y\Omega - C_2 = 0}}$

① $\Rightarrow$ $\omega_Z = \text{const}$ (fixed by IC)

The 'constants' C_1, C_2 have dimensions of velocity and are related to IC (see below)

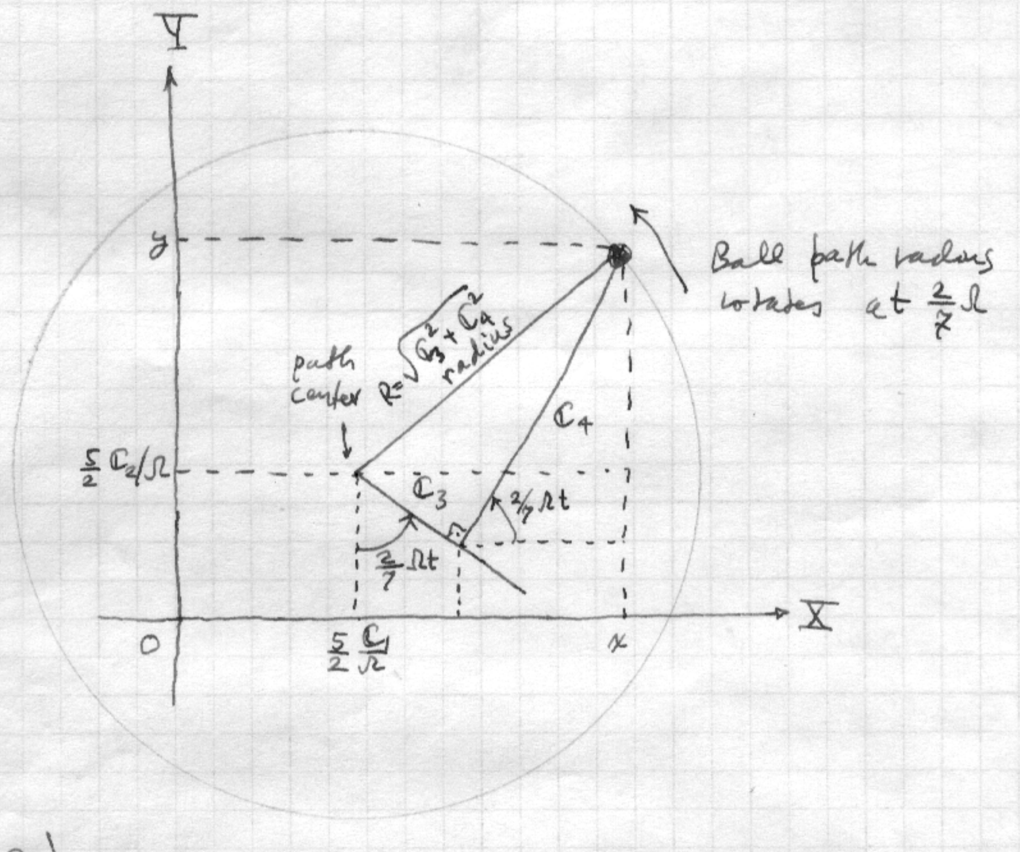
* Now, eliminating y $\Rightarrow$ $\ddot{x} + \left(\frac{2}{7}\Omega\right)^2 x = \frac{10}{49}\Omega C_1$

$$\left.\begin{aligned} \Rightarrow\quad x &= \frac{5}{2}\frac{C_1}{\Omega} + C_3 \sin\tfrac{2}{7}\Omega t + C_4 \cos\tfrac{2}{7}\Omega t \\ \text{So,}\quad y &= \frac{5}{2}\frac{C_2}{\Omega} - C_3 \cos\tfrac{2}{7}\Omega t + C_4 \sin\tfrac{2}{7}\Omega t \end{aligned}\right\}$$

'Constants' C_1, C_2, C_3, C_4 are fixed by IC $x|_0, y|_0, \dot{x}|_0, \dot{y}|_0$

* Path of C is always a <u>circle</u> of radius and center fixed by IC (see diagram); the rate at which ball traverses the circle is <u>fixed</u>, $\underline{\underline{\frac{2}{7}\Omega}}$

* As $\Omega \to 0$, radius $\to \infty$, ie, path approaches a straight line



$$\left.\begin{aligned}\left(x-\frac{5}{2}\frac{C_1}{\Omega}\right) &= C_3\sin\tfrac{2}{7}\Omega t + C_4\cos\tfrac{2}{7}\Omega t\\ \left(y-\frac{5}{2}\frac{C_2}{\Omega}\right) &= -C_3\cos\tfrac{2}{7}\Omega t + C_4\sin\tfrac{2}{7}\Omega t\end{aligned}\right\}$$

$$\left(x-\frac{5}{2}\frac{C_1}{\Omega}\right)^2+\left(y-\frac{5}{2}\frac{C_2}{\Omega}\right)^2 = C_3^2+C_4^2 = R^2$$

$$\left.\begin{aligned}x\big|_0 &= \frac{5}{2}\frac{C_1}{\Omega}+C_4, \quad y\big|_0 = \frac{5}{2}\frac{C_2}{\Omega}-C_3\\ \dot{x}\big|_0 &= \frac{2}{7}\Omega C_3, \quad \dot{y}\big|_0 = \frac{2}{7}\Omega C_4\end{aligned}\right\}\Rightarrow$$

$$C_3 = \frac{7}{2\Omega}\dot{x}\big|_0, \quad C_4 = \frac{7}{2\Omega}\dot{y}\big|_0, \quad C_1 = \frac{2}{5}\Omega x\big|_0 - \frac{7}{5}\dot{y}\big|_0$$

$$C_2 = \frac{2}{5}\Omega y\big|_0 + \frac{7}{5}\dot{x}\big|_0$$

<u>As $\Omega\to 0$</u> : $C_3, C_4 \to \infty \Rightarrow R\to\infty$ (radius $\to\infty$ ie, path approaches straight line)