MIT 2.111/8.411/6.898/18.435 Quantum Information Science I *Fall, 2010*
Sam Ocko September 9, 2010

Problem Set #1 Solutions

1. (a) The eigenvectors of

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Both of which have eigenvalue 1.

(b) The eigenvectors of

$$\sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are

$$\sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Having eigenvalues 1 and -1.

(c) The eigenvectors of

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

are

$$\sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Having eigenvalues 1 and -1.

(d) The eigenvectors of

$$\sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$$

Having eigenvalues 1 and -1.

2. The eigenvectors of the matrix

$$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

are

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \sqrt{\frac{1}{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \sqrt{\frac{1}{2}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix},$$

Having eigenvalues 1, 1, 1, -1

3. (a) $v^\dagger v = 1$
   (b) $v^\dagger w = 0$
   (c)
   $$vv^\dagger = \begin{bmatrix}1&0\\0&0\end{bmatrix}$$
   (d) $v^\dagger X w = 1$

4. (a) Let v be any normalized eigenvector of M with eigenvalue λ. $vv^\dagger = 1, Mv = \lambda v$
   $$\lambda = v^\dagger \lambda v = v^\dagger M v = \left(v^\dagger M^\dagger v\right)^\dagger = \left(v^\dagger M v\right)^\dagger = \lambda^\dagger = \lambda^*$$
   Therefore, λ is real-valued, and all eigenvalues of M are real.
   (b)
   $$v^\dagger M v = \left(v^\dagger M^\dagger v\right)^\dagger = \left(v^\dagger M v\right)^\dagger = \left(v^\dagger M v\right)^*$$
   Therefore, $v^\dagger M v$ is real-valued for all vectors v

5.

$$U = e^{iM} = \sum_k \frac{(iM)^k}{k!}$$

We then can say

$$U^\dagger = \sum_k \frac{\left((iM)^k\right)^\dagger}{k!} = \sum_k \frac{\left((iM)^\dagger\right)^k}{k!} = \sum_k \frac{(-iM)^k}{k!}$$

$$U^\dagger U = \sum_{j,k} \frac{(-iM)^j}{j!} \frac{(iM)^k}{k!}$$

Collecting terms, we obtain

$$U^\dagger U = \sum_{N=0}^{\infty} \sum_{k=0}^{N} \frac{(-iM)^{N-k}}{k!} \frac{(iM)^k}{k!} = \sum_{N=0}^{\infty} (-iM)^N \cdot \sum_{k=0}^{N} \frac{i^{2k}}{k!(N-k)!} = \sum_{N=0}^{\infty} (-iM)^N \cdot \sum_{k=0}^{N} \frac{(-1)^k}{k!(N-k)!}$$

When $N > 0$, we can use the Binomial Theorem to say:

$$\sum_{k=0}^{N} \frac{(-1)^k}{k!(N-k)!} = \frac{1}{N!}(1-1)^N = 0$$

therefore

$$U^\dagger U = (-iM)^0 = I$$

Alternate Proof An alternate proof relies on the fact that the eigenvectors of a Hermitian matrix are orthogonal. When M has eigenvectors v_j, eigenvalues λ_j we can write

$$M = \sum v_j \lambda_j v_j^\dagger, U = e^{iM} = \sum_j v_j e^{i\lambda_j} v_j^\dagger$$

$$U^\dagger U = \sum_j v_j e^{i\lambda_j} v_j^\dagger \sum_k v_k e^{-i\lambda_k} v_k^\dagger = I$$

Problem Set #1

Due: Thursday, September 17, 2010

1. **Eigenvalues and eigenvectors of the Pauli matrices**

   Give the eigenvectors and eigenvalues of these four matrices:

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. **Eigenvalues and eigenvectors of a** 4×4 **matrix**

   Give the eigenvalues and eigenvectors of this matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. **Inner products**

   For matrix M, let $M^\dagger = (M^T)^*$, where M^T is the transpose of M, and $*$ is denotes the complex conjugate of M. We call $M^\dagger$ the adjoint of M.

   Let

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   (a) What is $v^\dagger v$?

   (b) What is $v^\dagger w$?

   (c) What is $vv^\dagger$?

   (d) What is $v^\dagger X w$?

4. **Hermitian matrices**

   A matrix M is Hermitian if $M^\dagger = M$. Let M be Hermitian.

   (a) Prove that all of its eigenvalues are real.

   (b) Prove that $v^\dagger M v$ is real, for all vectors v. When $v^\dagger M v > 0$, we say that $M > 0$.

5. **Unitary matrices**

   Let M be Hermitian, and define

$$U = e^{iM} = \sum_k \frac{(iM)^k}{k!}$$

   Prove that $U^\dagger U = I$, where I is the identity matrix.