Problem Set #1

Due: Thursday, September 17, 2010

1. Eigenvalues and eigenvectors of the Pauli matrices
   Give the eigenvectors and eigenvalues of these four matrices:
   \[
   \sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
   \sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
   \]

2. Eigenvalues and eigenvectors of a 4×4 matrix
   Give the eigenvalues and eigenvectors of this matrix:
   \[
   \begin{bmatrix}
   1 & 0 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   0 & 1 & 0 & 0 \\
   0 & 0 & 0 & 1 
   \end{bmatrix}
   \]

3. Inner products
   For matrix \( M \), let \( M^\dagger = (M^T)^* \), where \( M^T \) is the transpose of \( M \), and \( * \) denotes the complex conjugate of \( M \). We call \( M^\dagger \) the adjoint of \( M \).
   Let
   \[
   v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
   \]
   (a) What is \( v^\dagger v \)?
   (b) What is \( v^\dagger w \)?
   (c) What is \( vv^\dagger \)?
   (d) What is \( v^\dagger Xw \)?

4. Hermitian matrices
   A matrix \( M \) is Hermitian if \( M^\dagger = M \). Let \( M \) be Hermitian.
   (a) Prove that all of its eigenvalues are real.
   (b) Prove that \( v^\dagger M v \) is real, for all vectors \( v \). When \( v^\dagger M v > 0 \), we say that \( M > 0 \).

5. Unitary matrices
   Let \( M \) be Hermitian, and define
   \[
   U = e^{iM} = \sum_k \frac{(iM)^k}{k!}
   \]
   Prove that \( U^\dagger U = I \), where \( I \) is the identity matrix.