Problem Set #1

Due: Thursday, September 17, 2010

1. **Eigenvalues and eigenvectors of the Pauli matrices**

   Give the eigenvectors and eigenvalues of these four matrices:

   $$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   $$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. **Eigenvalues and eigenvectors of a 4×4 matrix**

   Give the eigenvalues and eigenvectors of this matrix:

   $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. **Inner products**

   For matrix M, let $M^\dagger = (M^T)^*$, where M^T is the transpose of M, and $*$ is denotes the complex conjugate of M. We call $M^\dagger$ the adjoint of M.

   Let

   $$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   (a) What is $v^\dagger v$?
   (b) What is $v^\dagger w$?
   (c) What is $vv^\dagger$?
   (d) What is $v^\dagger X w$?

4. **Hermitian matrices**

   A matrix M is Hermitian if $M^\dagger = M$. Let M be Hermitian.

   (a) Prove that all of its eigenvalues are real.
   (b) Prove that $v^\dagger M v$ is real, for all vectors v. When $v^\dagger M v > 0$, we say that $M > 0$.

5. **Unitary matrices**

   Let M be Hermitian, and define

   $$U = e^{iM} = \sum_k \frac{(iM)^k}{k!}$$

   Prove that $U^\dagger U = I$, where I is the identity matrix.