1. **Density matrices.** A density matrix (also sometimes known as a density operator) is a representation of statistical mixtures of quantum states. This exercise introduces some examples of density matrices, and explores some of their properties.

(a) Let \( |\psi\rangle = a|0\rangle + b|1\rangle \) be a qubit state. Give the matrix \( \rho = |\psi\rangle \langle \psi| \), which you may compute using linear algebra using the vector representations of \( |\psi\rangle \) and \( \langle \psi| \). What are the eigenvectors and eigenvalues of \( \rho \)?

(b) Let \( \rho_0 = |0\rangle \langle 0| \) and \( \rho_1 = |1\rangle \langle 1| \). Give the matrix \( \sigma = \rho_0 + \rho_1 \). What are the eigenvectors and eigenvalues of \( \sigma \)?

(c) Compute \( \text{tr}(\rho^2) \) and \( \text{tr}(\sigma^2) \). In general, \( \text{tr}(M^2) \leq 1 \), with equality if and only if \( M \) is a pure state.

2. **Exponential of the Pauli matrices.** Let \( \vec{v} \) be any real, three-dimensional unit vector and \( \theta \) a real number. Prove that
\[
\exp(i\theta \vec{v} \cdot \vec{\sigma}) = \cos(\theta) I + i \sin(\theta) \vec{v} \cdot \vec{\sigma},
\]
where \( \vec{v} \cdot \vec{\sigma} \equiv \sum_{i=1}^{3} v_i \sigma_i \), and \( \sigma_i \) are the Pauli matrices, \( \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \).

3. **Hadamard operator on n qubits.** The Hadamard operator on one qubit may be written as
\[
H = \frac{1}{\sqrt{2}} \left[ (|0\rangle + |1\rangle)|0\rangle + (|0\rangle - |1\rangle)|1\rangle \right].
\]
Show explicitly that the Hadamard transform on \( n \) qubits, \( H^\otimes n \), may be written as
\[
H^\otimes n = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle \langle y|.
\]
Write out an explicit matrix representation for \( H^\otimes 2 \).

4. **Single qubit rotations.** Define the rotation operator
\[
R_\hat{n}(\theta) \equiv \exp(-i\theta \hat{n} \cdot \vec{\sigma}/2) = \cos \left( \frac{\theta}{2} \right) I - i \sin \left( \frac{\theta}{2} \right) (n_x X + n_y Y + n_z Z),
\]
where \( \hat{n} \) is a real three-dimensional unit vector.

1. Prove that an arbitrary single qubit unitary operator can be written in the form \( U = \exp(i\alpha) R_\hat{n}(\theta) \), for some real numbers \( \alpha \) and \( \theta \).

2. Find values for \( \alpha, \theta, \) and \( \hat{n} \) giving the Hadamard gate \( H \).

3. Find values for \( \alpha, \theta, \) and \( \hat{n} \) giving the phase gate
\[
S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.
\]