MASSACHUSETTS INSTITUE OF TECHNOLOGY

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Quantum Information Science I

September 16, 2010

Problem Set #2
(due in class, 23-Sep-10)

1. **Density matrices**. A density matrix (also sometimes known as a *density operator*) is a representation of statistical mixtures of quantum states. This exercise introduces some examples of density matrices, and explores some of their properties.

   (a) Let $|\psi\rangle = a|0\rangle + b|1\rangle$ be a qubit state. Give the matrix $\rho = |\psi\rangle\langle\psi|$, which you may compute using linear algebra using the vector representations of $|\psi\rangle$ and $\langle\psi|$. What are the eigenvectors and eigenvalues of ρ?

   (b) Let $\rho_0 = |0\rangle\langle 0|$ and $\rho_1 = |1\rangle\langle 1|$. Give the matrix $\sigma = \frac{\rho_0+\rho_1}{2}$. What are the eigenvectors and eigenvalues of σ?

   (c) Compute $\mathrm{tr}(\rho^2)$ and $\mathrm{tr}(\sigma^2)$. In general, $\mathrm{tr}(M^2) \leq 1$, with equality if and only if M is a pure state.

2. **Exponential of the Pauli matrices**. Let $\vec{v}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp(i\theta\vec{v}\cdot\vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}, \tag{1}$$

   where $\vec{v}\cdot\vec{\sigma} \equiv \sum_{i=1}^{3} v_i\sigma_i$, and σ_i are the Pauli matrices, $\sigma_1 = X$, $\sigma_2 = Y$, $\sigma_3 = Z$.

3. **Hadamard operator on n qubits**. The Hadamard operator on one qubit may be written as

$$H = \frac{1}{\sqrt{2}}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big]. \tag{2}$$

   Show explicitly that the Hadamard transform on n qubits, $H^{\otimes n}$, may be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y|. \tag{3}$$

   Write out an explicit matrix representation for $H^{\otimes 2}$.

4. **Single qubit rotations**. Define the rotation operator

$$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\,\hat{n}\cdot\vec{\sigma}/2) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)(n_xX + n_yY + n_zZ), \tag{4}$$

   where $\hat{n}$ is a real three-dimensional unit vector.

   1. Prove that an arbitrary single qubit unitary operator can be written in the form $U = \exp(i\alpha)R_{\hat{n}}(\theta)$, for some real numbers α and θ.
   2. Find values for α, θ, and $\hat{n}$ giving the Hadamard gate H.
   3. Find values for α, θ, and $\hat{n}$ giving the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}. \tag{5}$$