Problem Set # 3 Solutions

1. Measurements and Uncertainty

(a) The expectation value of $M$ on state $|\psi\rangle$ will be $m$, the standard deviation will be 0.
(b) Measuring $X$ on the state $|0\rangle$, we will get results of 1,-1 with equal probability. Therefore the expectation value is 0 and the standard deviation is 1.

2. Entropy of quantum states

(a) The entropy of $\rho_0 = |0\rangle \langle 0|$ is $-\log_2(1) = 0$
(b) The entropy of $\rho_1 = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| + \langle 1| \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{1}{2} \log_2(\frac{1}{2}) = 1$
(c) If $Tr(\rho^2) = 1$,
\[ \sum_k \lambda_k^2 = \sum_k \lambda_k = 1 \]
Therefore,
\[ \sum_k \lambda_k (\lambda_k - 1) = 0 \]
Since $0 \leq \lambda_k \leq 1$, $\forall k$, we know that $\lambda_k (\lambda_k - 1) \geq 0$, $\forall k$, and thus the only way for the above condition to be satisfied is for $\lambda_k = 0, 1$, $\forall k$. Therefore $Tr(\rho^2) = 1$ if and only if $\rho$ has a single eigenvalue of 1 with all other eigenvalues 0.

\[ S(\rho) = -\sum_k \lambda_k \log_2(\lambda_k) = 0 \]
Since $0 \leq \lambda_k \leq 1$, $\forall k$, we know that $\lambda_k \log_2(\lambda_k) \geq 0$, $\forall k$. Therefore, the only way for the above condition to be satisfied is for $\lambda_k = 0, 1$, $\forall k$, and thus $S(\rho) = 1$ if and only if $\rho$ has a single eigenvalue of 1 with all other eigenvalues 0.

Therefore, for density matrices, $Tr(\rho^2) = 1$ and $S(\rho) = 1$ are equivalent statements.

3. (a) A state is a product state if and only if it can be represented as $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. If a state has a Schmidt number 1, it can be represented as a product state $\sum_k \sqrt{\lambda_k} |k_A\rangle |k_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ since only one Schmidt coefficient is nonzero. If it has a Schmidt number greater than 1, it has no such representation as $|\psi_A\rangle \otimes |\psi_B\rangle$, because if it did it would have a Schmidt number of 1 through the above representation.
(b) Lemma: If an entangled state between Alice and Bob has the Schmidt decomposition

\[ \sum_k \sqrt{\lambda_k} |k_A\rangle |k_B\rangle \]

Then Alice’s reduced density matrix is

\[ \rho^A = \sum_k \lambda_k |k_A\rangle \langle k_A| \]

(Likewise for Bob)

Therefore, if \(|\psi\rangle\) has a Schmidt number of 1, the reduced density matrices \(\rho^A, \rho^B\) have only one non-zero eigenvalue and are pure states. If \(|\psi\rangle\) has a Schmidt number greater than 1, the reduced density matrices \(\rho^A, \rho^B\) have multiple non-zero eigenvalues and are mixed states.

Proof Of Lemma (Less Mathematical): If Bob measured his state in the \(\{|k_B\rangle\}\) basis, with probability \(\lambda_k\) he will measure \(|k_B\rangle\) and Alice’s state will collapse to \(|k_A\rangle\). Therefore, since the outcome of Alice’s measurement can’t be effected by whether Bob made his measurement, we may say:

\[ \rho^A = \sum_k \lambda_k |k_A\rangle \langle k_A| \]

Proof of Lemma (More Mathematical): We may write the global density matrix as

\[ \rho^{total} = \sum_{k,K'} \sqrt{\lambda_k \lambda_{k'}} |k_A\rangle_A |k_B\rangle_B \langle k'_B|_B \langle k'_A|_A \]

Alice’s reduced density matrix can be obtained by taking the partial trace

\[ \rho^A = Tr_B (\rho^{total}) = \sum_{k,K'} \sqrt{\lambda_k \lambda_{k'}} |k_A\rangle \langle k_A| \langle k'_A| \langle k'_B|_B \]

Since \( Tr_2 (|k_B\rangle \langle k_B|) = \langle k'_B|_B |k_B\rangle = \delta_{kk'} \), we may say

\[ \rho^A = \sum_k \lambda_k |k_A\rangle \langle k_A| \]

4. (a) The Schmidt decomposition of \(|\phi_1\rangle = \frac{00 + 11 + 22}{\sqrt{3}}\) is \(\sum_{k=0,1,2} \frac{1}{\sqrt{3}} |k\rangle |k\rangle\) by inspection (Other Schmidt decompositions are also possible)

(b) The Schmidt decomposition of \(|\phi_2\rangle = \frac{00 + 01 + 10 + 11}{2}\) is \(|+\rangle \langle +|\) by inspection, where \(|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\)

(c) The Schmidt decomposition of \(|\phi_3\rangle = \frac{00 + 01 + 10 - 11}{\sqrt{3}}\) is \(\frac{1}{\sqrt{2}} (|0\rangle \langle +| + |1\rangle \langle -|)\) by inspection, where \(|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)\) (Other Schmidt decompositions are also possible)

(d) To find the schmidt decomposition of \(|\phi_4\rangle = \frac{00 + 01 + 11}{\sqrt{3}}\), we use the lemma proved in the previous problem. We can see that Alice’s reduced density matrix is

\[ \rho^A = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \]
having eigenvalues $\frac{1}{6} (3 + \sqrt{5}) \approx .87, \frac{1}{6} (3 - \sqrt{5}) \approx .13$, eigenvectors

$$\frac{(-1 + \frac{1}{2} (3 + \sqrt{5}), 1)}{\sqrt{1 + (-1 + \frac{1}{2} (3 + \sqrt{5}))^2}} \approx (.85, .52), \frac{(-1 + \frac{1}{2} (3 - \sqrt{5}), 1)}{\sqrt{1 + (-1 + \frac{1}{2} (3 - \sqrt{5}))^2}} \approx (-.52, .85)$$

Likewise, Bob’s reduced density matrix is

$$\rho_B = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

having eigenvalues $\frac{1}{6} (3 + \sqrt{5}) \approx .87, \frac{1}{6} (3 - \sqrt{5}) \approx .13$, eigenvectors

$$\frac{(1, -1 + \frac{1}{2} (3 + \sqrt{5}))}{\sqrt{1 + (-1 + \frac{1}{2} (3 + \sqrt{5}))^2}} \approx (.52, .85), \frac{(-1, 1 - \frac{1}{2} (3 - \sqrt{5}))}{\sqrt{1 + (-1 + \frac{1}{2} (3 - \sqrt{5}))^2}} \approx (-.85, .52)$$

Therefore, the Schmidt decomposition will be

$$|\phi_4\rangle = \frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} = \sqrt{\lambda_1} |\psi_{A1}\rangle |\psi_{B1}\rangle + \sqrt{\lambda_2} |\psi_{A2}\rangle |\psi_{B2}\rangle$$

Where $\lambda_1 = \frac{1}{6} (3 + \sqrt{5}) \approx .87, \lambda_2 = \frac{1}{6} (3 - \sqrt{5}) \approx .13$

And the Schmidt vectors $|\psi_{A1}\rangle, |\psi_{A2}\rangle, |\psi_{B1}\rangle, |\psi_{B2}\rangle$ are defined as above

5. (a) $\psi_1 = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$
   $\psi_2 = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi} |1\rangle)$
   $\psi_3 = \frac{1}{2} ((1 + e^{i\phi}) |0\rangle + (1 - e^{i\phi}) |1\rangle)$

   (b) The probability of measuring a qubit to be 1 is $p = \frac{1}{4} |1 - e^{i\phi}|^2 = \frac{1 - \cos(\phi)}{2}$

   (c) We consider the random variable $X$ which is 0 if we measure 0 and 1 if we measure 1.

   $$\langle X \rangle = \langle X^2 \rangle = \frac{1 - \cos(\phi)}{2}$$

   Therefore, the variance of a single measurement is

   $$\langle X^2 \rangle - \langle X \rangle^2 = \frac{1 - \cos(\phi)}{2} - \left( \frac{1 - \cos(\phi)}{2} \right)^2 = \frac{\sin(\phi)^2}{4}$$

   Thus the variance in the number of 1’s you get after $n$ measurements is $n \frac{\sin(\phi)^2}{4}$

   and thus the standard deviation of measured probability after $n$ experiments will be

   $$\frac{|\sin(\phi)|}{\sqrt{n}}$$

   Since $\frac{dp}{d\phi} = \frac{\sin(\phi)}{2}$, the accuracy of the estimate for $\phi$ is

   $$\Delta \phi = \frac{\Delta p}{\frac{dp}{d\phi}} = \frac{\frac{\sin(\phi)}{2}}{\frac{\sin(\phi)}{\sqrt{n}}} = \frac{1}{\sqrt{n}}$$

   (Note that it’s impossible to tell the sign of $\phi$ in this way, you need to measure in a different basis to do that)