Problem Set #8
(due in class, TUESDAY 16-Nov-10)

P1: (Universality of Heisenberg Hamiltonian) Consider the two-qubit Heisenberg Hamiltonian
\[ H(t) = J(t) \hat{S}_1 \cdot \hat{S}_2 = \frac{J(t)}{4} \left[ X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 \right]. \] (1)

(a) Show that a swap operation \( U \) can be implemented by turning on \( J(t) \) for an appropriate amount of time \( \tau \), to obtain \( U = \exp(-i\pi \hat{S}_1 \cdot \hat{S}_2) \).

(b) Compute the \( \sqrt{\text{SWAP}} \) gate, obtained by turning on \( J(t) \) for time \( \tau/2 \). Together with arbitrary single qubit gates, the \( \sqrt{\text{SWAP}} \) gate is universal for quantum computation.

P2: (NMR controlled-\text{\textbf{NOT}}) Give an explicit sequence of single qubit rotations which realize a controlled-\text{\textbf{NOT}} between two spins evolving under the Hamiltonian \( H = aZ_1 + bZ_2 + cZ_1Z_2 \). Simplify the result as much as you can, to reduce the number of single qubit rotations.

P3: (Simple harmonic oscillators) Consider the Hamiltonian \( H = \hbar \omega a^\dagger a \), and let \( |n\rangle \) be an energy eigenstate with energy \( n\hbar\omega \).

(a) Compute \([H, a] = Ha - aH\) and use the result to show that if \( |\psi\rangle \) is an eigenstate of \( H \) with energy \( E \geq n\hbar\omega \), then \( a^n|\psi\rangle \) is an eigenstate with energy \( E - n\hbar\omega \).

(b) Show that \( |n\rangle = \frac{(a\dagger)^n}{\sqrt{n!}} |0\rangle \).

P4: (Coherent states) The coherent state \( |\alpha\rangle \) of a simple harmonic oscillator is defined as
\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \] (2)

where \( |n\rangle \) is an \( n \)-photon energy eigenstate.

(a) Prove that a coherent state is an eigenstate of the photon annihilation operator, that is, show \( a|\alpha\rangle = \lambda|\alpha\rangle \) for some constant \( \lambda \).

(b) Compute \( \langle \alpha|\alpha \rangle \).

(c) Let \( B = \exp \left[ \theta \left( a^\dagger b - ab^\dagger \right) \right] \) be the beamsplitter operator coupling two simple harmonic oscillators. Using the fact that \( BaB^\dagger = a\cos\theta + b\sin\theta \) and \( BbB^\dagger = -a\sin\theta + b\cos\theta \), compute \( B|\alpha\rangle|\beta\rangle \) where \( |\alpha\rangle \) and \( |\beta\rangle \) are coherent states of the two systems. Express the result in a simple form.

P5: (Recent implementations of quantum algorithms and protocols) Find a recent paper in the literature about an implementation of a quantum algorithm (preferably) or a quantum protocol, involving more than one qubit. Write a short (< 500 word) summary of it, on the QIS wiki. See instructions on the course homepage, \text{http://web.mit.edu/2.111/}