2.111J/18.435J Quantum Computation Final Exam Solutions
(Given 9 AM – 12 Noon, with 2 pages of notes allowed, on Wednesday, December 21, 2005)

(0) What Is a Tensor? (*9 points*)

Solution: With the requisite apologies to Emily Dickinson for turning her famous poem #254 (" 'Hope' is the thing with feathers. . . ") into a pedagogical jingle,

'Tensor' is the thing with slots —
that perches on the vector. . .

More seriously, a tensor is a multilinear map on a composition of vector spaces. That is, a tensor is the composition of linear operators $\{O_1, O_2, \ldots\}$ that respectively act on vector spaces $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$. A tensor thus takes as its input a composition of vectors $\{|v_1\rangle \in \mathcal{H}_1, |v_2\rangle \in \mathcal{H}_2, \ldots\}$ and outputs another composition of vectors. These compositions are generally denoted with the symbol $\otimes$, the "tensor product"

$$(O_1 \otimes O_2 \otimes \ldots \otimes O_n)(|v_1\rangle \otimes |v_2\rangle \otimes \ldots \otimes |v_n\rangle) = O_1|v_1\rangle \otimes O_2|v_2\rangle \otimes \ldots \otimes O_n|v_n\rangle.$$

(1) Buzzwords. Fill in the blank with appropriate word or phrase. (*2 points apiece*)

(1a) 'Harry and are perfect for each other. We seem to know more about each other's feelings than is humanly possible. We're like two ____________ qubits.'

(1b) 'I'm so confused, I can't figure out which direction is up and which is down. I'm like a nuclear spin in a coherent ____________ of $|0\rangle$ and $|1\rangle$.'

(1c) A ____________ gate and a ____________ gate are both reversible logic gates. The first preserves the number of zeros and ones in the input, the second does not.

(1d) 'Jeanette is so logical, it can be frustrating. Sometimes I think that she wants to reduce our whole relationship to a calculation in ____________ ____________ .'

(1e) The ____________ state and the ____________ state both exhibit quantum weirdness. But the former has only two qubits while the latter has three.

(1f) Quantum computation poses a threat to conventional ____________ cryptography.

(1g) 'You and I are looking at this relationship from completely complementary points of view. It's as if I'm position and you're momentum. If we shared a bunch of qubits, for you to see things my way, you would have to perform a ____________ ____________ ____________ on them.'

(1h) A nuclear spin can't do a belly-flop, but it can do a ____________ ____________.

Solutions:

(1a) entangled

(1b) superposition

(1c) SWAP, CNOT (to take perhaps the 2 best known examples)

(1d) NP COMPLETE

(1e) Bell (or EPR, for Einstein-Podolsky-Rosen), GHZ (for Greenberger-Horne-Zeilinger)

(1f) Public-key (or RSA, for Rivest-Shamir-Adleman)

(1g) quantum fourier transform

(1h) Rabi flop

(2) Magnetic Resonance. (*25 points*)

A proton is sitting in a static magnetic field of 10.000 Tesla, oriented along the z-axis. You can apply an oscillating microwave field to the proton. The strength of the oscillating field is 0.100 Tesla. You can choose both the frequency of the applied field and the duration of the microwave pulse to be applied. The proton is initially oriented with its spin up along the z-axis. Its magnetic moment is $1.410606633 \times 10^{-26}$ Joules/Tesla. (Another fundamental constant you may wish to note is $\hbar = 1.054573 \times 10^{-34}$ Joules $\cdot$ sec.)

(2a) Your goal is to rotate the proton spin by an angle π about the axis $(\hat{x} + \hat{z})/\sqrt{2}$ in the co-rotating frame. This action can be performed by applying a single microwave pulse. What frequency should the pulse have?

(2b) For what duration should the pulse be applied?

(2c) Discuss how the effect of the pulse applied in (2a-b) resembles and differs from an idealized Hadamard gate.

Solution:

The Hamiltonian for a magnetic dipole $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}(t)$ is $H(t) = -\boldsymbol{\mu} \cdot \mathbf{B}(t)$. The quantum operator for the magnetic dipole moment is directly proportional to the spin operator. In our case of the proton, $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z) = k(\sigma_x, \sigma_y, \sigma_z)$ where the constant of proportionality $k = 1.410606633 \times 10^{-26}$ Joules/Tesla, as given above. In the Rabi case, our oscillating microwave field is circularly polarized and therefore our magnetic field is $\mathbf{B}(t) = (B_\perp \cos \omega t,\ B_\perp \sin \omega t,\ B_z)$ where $B_\perp = 0.100$ Tesla and $B_z = 10.000$ Tesla. So, in short, we are tasked to solve Schrödinger's Equation for the Hamiltonian

$$H(t) = -k\Big(B_z \sigma_z + B_\perp \big[(\cos \omega t)\sigma_x + (\sin \omega t)\sigma_y\big]\Big).$$

We solve this by going to the coordinate frame that co-rotates with the oscillating field.

Thus, we have the coordinate frame rotate with angle ωt around the z-axis. Thus, we must rotate the state by an angle $-\omega t$ around the z-axis

$$|\psi_{\text{rot}}(t)\rangle = e^{i\omega t\sigma_z/2}|\psi(t)\rangle$$

[*Note on Sign Conventions*: The angular frequency ω can be either positive or negative. Positive (negative) rotation angles denote counterclockwise (clockwise) rotations.]

We rewrite Schrödinger's Equation in terms of $|\psi_{\text{rot}}(t)\rangle$ as follows:

$$\begin{aligned}
\frac{\partial|\psi(t)\rangle}{\partial t} &= -\frac{i}{\hbar}H(t)|\psi(t)\rangle \\
\frac{\partial\big(e^{-i\omega t\sigma_z/2}|\psi_{\text{rot}}(t)\rangle\big)}{\partial t} &= -\frac{ik}{\hbar}\Big(B_z\sigma_z + B_\perp\big[(\cos\omega t)\sigma_x + (\sin\omega t)\sigma_y\big]\Big)e^{-i\omega t\sigma_z/2}|\psi_{\text{rot}}(t)\rangle \\
e^{-i\omega t\sigma_z/2}\left(-\frac{i\omega}{2}\sigma_z + \frac{\partial}{\partial t}\right)|\psi_{\text{rot}}(t)\rangle &= -\frac{ik}{\hbar}\Big(B_z\sigma_z + B_\perp\big[e^{-i\omega t}\sigma_+ + e^{i\omega t}\sigma_-\big]\Big)e^{-i\omega t\sigma_z/2}|\psi_{\text{rot}}(t)\rangle
\end{aligned}$$

thus upon pre-multiplying both sides by $e^{i\omega t\sigma_z/2}$ and rearranging, we calculate

$$\begin{aligned}
\frac{\partial|\psi_{\text{rot}}(t)\rangle}{\partial t} &= \left[\frac{i\omega}{2}\sigma_z - e^{i\omega t\sigma_z/2}\left(\frac{ikB_z}{\hbar}\sigma_z + \frac{ikB_\perp}{\hbar}\big[e^{-i\omega t}\sigma_+ + e^{i\omega t}\sigma_-\big]\right)e^{-i\omega t\sigma_z/2}\right]|\psi_{\text{rot}}(t)\rangle \\
&= -\frac{i}{\hbar}\left[\left(kB_z - \frac{\hbar\omega}{2}\right)\sigma_z + kB_\perp\sigma_x\right]|\psi_{\text{rot}}(t)\rangle,
\end{aligned}$$

thus arriving at our desired time-independent Schrödinger's Equation

$$\frac{\partial|\psi_{\text{rot}}(t)\rangle}{\partial t} = -\frac{i}{\hbar}H_{\text{rot}}|\psi_{\text{rot}}(t)\rangle \text{ where } H_{\text{rot}} \equiv \left(kB_z - \frac{\hbar\omega}{2}\right)\sigma_z + kB_\perp\sigma_x.$$

Noting that we may rewrite H_{rot} as

$$H_{\text{rot}} \equiv E\left[\frac{1}{E}\left(kB_z - \frac{\hbar\omega}{2}\right)\sigma_z + \frac{kB_\perp}{E}\sigma_x\right] \text{ where } E = \sqrt{\left(kB_z - \frac{\hbar\omega}{2}\right)^2 + k^2B_\perp^2}$$

and noting that the matrix in large brackets is simply the Pauli matrix $\sigma_{\hat{n}}$ for the axis

$$\hat{n} = \left(\frac{kB_\perp}{E},\, 0,\, \frac{kB_z}{E} - \frac{\hbar\omega}{2E}\right)$$

and noting that a rotation of an angle θ around an axis $\hat{n}$ is encoded by $e^{-i\theta\sigma_{\hat{n}}/2}$, we conclude that:

> *In the co-rotating frame, a magnetic field* $\mathbf{B}(t) = (B_\perp\cos\omega t,\, B_\perp\sin\omega t,\, B_z)$ *drives the dipole state to precess around the axis* $\hat{n} \propto (kB_\perp,\, 0,\, kB_z - \hbar\omega/2)$ *at an angular frequency of*
>
> $$\Omega_{\text{Rabi}}(\omega) = \frac{2E}{\hbar} = \frac{2}{\hbar}\sqrt{\left(kB_z - \frac{\hbar\omega}{2}\right)^2 + k^2B_\perp^2}$$

With this result in hand, we may now finally answer (2a-c).

(2a) What ω produces a rotation axis of $(\hat{x}+\hat{z})/\sqrt{2}$?

Answer: We need the ω such that $kB_\perp = kB_z - \hbar\omega/2$, thus we need

$$\omega = \frac{2k(B_\perp - B_z)}{\hbar} = \frac{2(1.410607\times10^{-26}\ \text{Joules/Tesla})(-0.900\ \text{Tesla})}{1.054573\times10^{-34}\ \text{Joules}\cdot\text{sec}} = -240.7697\ \text{MHz}$$

That is, to the 3 significant digits we actually have, we need a magnetic field rotating *clockwise* with an angular frequency of 241 MHz.

(2b) For what duration should the pulse be applied?

Answer: We desire a duration t such that $t\Omega_{Rabi}(\omega) = \pi$ with the ω of (2a). Thus we need

$$t = \frac{\hbar\pi}{2}\left[\left(kB_z - \frac{\hbar\omega}{2}\right)^2 + k^2B_\perp^2\right]^{-1/2}$$

We note that

$$\begin{aligned}
\frac{\hbar\pi}{2} &= \frac{(1.054573\times10^{-34}\ \text{Joules}\cdot\text{sec})(3.141592654)}{2} = 1.656519\times10^{-34}\ \text{Joules}\cdot\text{sec}\\
\frac{\hbar\omega}{2} &= \frac{(1.054573\times10^{-34}\ \text{Joules}\cdot\text{sec})(2.407697\times10^{8}\ \text{Hz})}{2} = 1.269546\times10^{-26}\ \text{Joules}\\
kB_z &= (1.410607\times10^{-26}\ \text{Joules/Tesla})(10.000\ \text{Tesla}) = 1.410607\times10^{-25}\ \text{Joules}\\
kB_x &= (1.410607\times10^{-26}\ \text{Joules/Tesla})(0.100\ \text{Tesla}) = 1.410607\times10^{-27}\ \text{Joules}
\end{aligned}$$

Thus,

$$\begin{aligned}
t &= (1.656519\times10^{-34}\text{Joules}\cdot\text{sec})\left[\left(1.283652\times10^{-25}\text{Joules}\right)^2 + \left(1.410607\times10^{-27}\text{Joules}\right)^2\right]^{-1/2}\\
&= 1.29\ \text{nanoseconds}
\end{aligned}$$

(2c) The time evolution differs by nontrivial (*i.e.*, "relative" or state-specific) phase factors from an ideal Hadamard gate $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$. Specifically, with the parameters found in (2a-b), the unitary time evolution in the co-rotating frame is

$$U_{rot} = e^{-i\pi(\sigma_z+\sigma_x)/(2\sqrt{2})} = \cos\left(\frac{\pi}{2}\right)\mathbb{I} - \frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2}\right)(\sigma_z+\sigma_x) = -\frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

Therefore, in the original frame, the unitary time evolution is

$$U = e^{-i\omega t\sigma_z/2}U_{\text{rot}} = -\frac{i}{\sqrt{2}}\begin{pmatrix}e^{-i\omega t/2} & e^{-i\omega t/2}\\ e^{i\omega t/2} & -e^{i\omega t/2}\end{pmatrix}.$$

(3) Josephson Junction Hamiltonian. (*25 points*)

The Hamiltonian for a Josephson junction system is $H = P^2/2 - \cos X$. Here, P and X are operators that are analogous to momentum and position: X correponds to the phase Φ of the supercurrent across the junction, and P corresponds to $d\Phi/dt$. Note that in the vicinity of $|x = 0\rangle$, this Hamiltonian is close to that of an harmonic oscillator. As x moves away from 0, the operator becomes anharmonic. The Hamiltonian H is in fact the Hamiltonian for a *physical pendulum*, in other words, a rigid rotor acted on by a constant force.

(3a) Using the commutation relation, $[X, P] = i\hbar$, find formulae for $[X, P^2]$ and $[P, X^2]$.

(3b) Find formulae for $[X, P^n]$ and $[P, X^n]$.

(3c) In the Heisenberg picture of quantum mechanics, one looks at the time evolution of *operators* such as X and P, rather than the time evolution of states such as $|x\rangle$ and $|p\rangle$. The time evolution of an operator A is given by $dA/dt = (i/\hbar)[H, A]$. Using the results of (3a-b), give expressions for dX/dt and dP/dt.

Solutions:

(3a) $[X, P^2] = XPP - PPX = XPP - P(XP - [X, P]) = [X, P]P + i\hbar P = 2i\hbar P$

$$[P, X^2] = PXX - XXP = PXX - X(PX + [X, P]) = [P, X]X - i\hbar X = -2i\hbar X$$

(3b) Similarly to (3a), we calculate

$$\begin{aligned}
[X, P^n] &= XP^n - P^n X \\
&= XP^n - P^{n-1}(XP - [X, P]) \\
&= [X, P^{n-1}]P + i\hbar P^{n-1}
\end{aligned}$$

We then iterate this formula.

$$\begin{aligned}
[X, P^n] &= [X, P^{n-1}] + i\hbar P^{n-1} \\
&= \big([X, P^{n-2}] + i\hbar P^{n-2}\big)P + i\hbar P^{n-1} \\
&= [X, P^{n-2}]P^2 + 2i\hbar P^{n-1} \\
&\vdots \\
&= [X, P^{n-k}]P^k + ki\hbar P^{n-1} \\
&\vdots \\
&= [X, P]P^{n-1} + (n-1)i\hbar P^{n-1} \\
&= ni\hbar P^{n-1}
\end{aligned}$$

Completely analogously, we find

$$
\begin{aligned}
[P, X^n] &= PX^n - X^n P \\
&= PX^n - X^{n-1}\big(PX + [X,P]\big) \\
&= [P, X^{n-1}]X - i\hbar X^{n-1} \\
&= \big([P, X^{n-2}] - i\hbar X P^{n-2}\big)X - i\hbar X^{n-1} \\
&= [P, X^{n-2}]X^2 - 2i\hbar X^{n-1} \\
&\vdots \\
&= [P, X^{n-k}]X^k - ki\hbar X^{n-1} \\
&\vdots \\
&= [P, X]X^{n-1} - (n-1)i\hbar X^{n-1} \\
&= -ni\hbar X^{n-1}
\end{aligned}
$$

(3c) Evaluating $dX/dt = (i/\hbar)[H, X]$ is readily done just with the result $[X, P^2] = 2i\hbar P$ from (3a) since $[X, \cos X] = 0$. Explicitly,

$$
\frac{dX}{dt} = \frac{i}{\hbar}[H, X] = \frac{i}{2\hbar}[P^2, X] = \frac{i}{2\hbar}(-2i\hbar P) = P
$$

as one would expect classically for a particle of unit mass. (NB: This should not be a surprise. Heisenberg's "matrix mechanics" essentially is the trick of simply replacing the Poisson brackets of classical Hamiltonian mechanics with commutators and observing the consequences.) Evaluating $dP/dt = (i/\hbar)[H, P] = (i/\hbar)[\cos X, P]$ requires two other facts. First, we need the Taylor expansion of $\cos X$ around $X = 0$,

$$
\cos X = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n}}{(2n)!} = \mathbb{I} - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \ldots
$$

Second, we need the formula for $[P, X^n] = -ni\hbar X^{n-1}$ from (3b). Combining these 2 facts, we conclude

$$
\begin{aligned}
\frac{dP}{dt} &= \frac{i}{\hbar}[\cos X, P] \\
&= \frac{i}{\hbar}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}[X^{2n}, P] \\
&= \frac{i}{\hbar}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\big(2ni\hbar X^{2n-1}\big) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} X^{2n-1}}{(2n-1)!} \\
&= \sin X
\end{aligned}
$$

which again is in accord with the classical mechanical result $dp/dt = -dV/dx = -dH/dx$.

(4) Decoherence Free Subspace. (*25 points*)

A collective error is one in which the same error occurs to each qubit. For example, if there are four qubits, a collective error could be of the form $U \otimes U \otimes U \otimes U$, where $U = e^{-i\theta\sigma/2}$ is a rotation by θ about an axis determined by σ. A four qubit quantum code to correct collective errors is defined as follows:

$$|0_C\rangle \to \frac{1}{2}(|01\rangle - |10\rangle) \otimes (|01\rangle - |10\rangle)$$

$$|1_C\rangle \to \frac{1}{\sqrt{12}}(2|1100\rangle + 2|0011\rangle - |1010\rangle - |0101\rangle - |0110\rangle - |1001\rangle).$$

(4a) Calculate the effect of an error $U \otimes U \otimes U \otimes U$ on the encoded version of the state $\alpha|0\rangle + \beta|1\rangle$, first for $U = \sigma_z$ and then for $U = e^{-i\theta\sigma_z/2}$. Show your work.

(4b) Calculate the effect of $U \otimes U \otimes U \otimes U$, where $U = e^{-i\theta\sigma_x/2}$.

(4c) Calculate the effect of $U \otimes U \otimes U \otimes U$, where $U = e^{-i\theta\sigma_y/2}$.

(4d) What is the effect of $U \otimes U \otimes U \otimes U$, for a generic $U = e^{-i\theta\sigma/2}$?

Solution:

It turns out that $U \otimes U \otimes U \otimes U$ with a generic $U = e^{-i\theta\sigma/2}$ will leave unchanged an arbitrary superposition of the codewords $\alpha|0_C\rangle + \beta|1_C\rangle$.

The key to proving this and thus answering (a-d) is to realize that $(\sigma \otimes \sigma \otimes \sigma \otimes \sigma)^2 = \mathbb{I}$ since $\sigma^2 = \mathbb{I}$. This allows us to invoke the theorem that for any matrix M such that $M^2 = \mathbb{I}$ and any complex scalar α, $e^{i\alpha M} = (\cos\alpha)\mathbb{I} + i(\sin\alpha)M$.

Thus, for us, the pertinent corollary is

$$e^{-i\theta(\sigma\otimes\sigma\otimes\sigma\otimes\sigma)/2} = (\cos\theta)\mathbb{I} - i(\sin\theta)(\sigma \otimes \sigma \otimes \sigma \otimes \sigma)$$

With this corollary in hand, readily show that an arbitrary superposition of the codewords $\alpha|0_C\rangle + \beta|1_C\rangle$ is invariant by any of the $U \otimes U \otimes U \otimes U$ in (4a-c). The reason is that the codewords are invariant under

- $\mathbb{I}$ (*Reason*: true by definition)
- $\sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$ (*Reason*: In $|0_C\rangle$, we have two singlets and thus the phase flips that occur in each singlet when every qubit is phase flipped will cancel each other out. In $|1_C\rangle$, every ket has an even number of 1's, making the resulting phase flips—which only occur on 1's—occur in pairs that cancel out.)

- $\sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$ (*Reason*: In $|0_C\rangle$, we have two singlets and thus the phase flips that occur in each singlet when every qubit is bit flipped will cancel each other out. In $|1_C\rangle$, all pairs of kets mapped to one another by a bit flip on each qubit always have the same coefficient in front of them.)
- $\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$ (*Reason*: We note that $\sigma_y = -i\sigma_z\sigma_x$. Thus, the reasons given above for invariance of codewords in the σ_z and σ_x cases imply invariance for this case.)

As for the generic case in (4d), we need but one more fact, which is that one can rewrite an arbitrary one-qubit rotation $e^{-i\theta\sigma/2}$ as the product of two one-qubit rotations around linearly independent axes, say $e^{-i\alpha\sigma_z/2}e^{-i\beta\sigma_x/2}$, for appropriately chosen α and β.