2.111J/18.435J Quantum Computation Midterm Quiz Solutions

(Given in class, with a page of notes allowed, on Thursday, October 27, 2005)

1) Classical Logic

Consider a binary function f from n bits to m bits. If one can write $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$, where $\mathbf{x}$ is an column vector of length n, M is a $m \times n$ matrix, and b is a column vector of length m, then f is said to be *affine*. (All multiplication and addition in this problem is mod 2.)

(a) Let $M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Exhibit a circuit made only of CNOT gates that implements $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$. (Your circuit can include more than 3 bits.)

(b) Is f one-to-one? How is your answer to this question reflected in the form of your circuit in (a)?

(c) Can any affine function f from n bits to m bits be constructed out of CNOT gates alone? (You are free to include more than $\max\{m, n\}$ bits in these hypothetical circuits.) If so, why? If not, why not?

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2) Single Qubit Operators

Note on Conventions: In this problem, positive rotation angles correspond to counterclockwise rotations, and negative rotation angles correspond to clockwise rotations.

(a) A qubit is rotated by $\pi/2$ around the x-axis and then rotated by $\pi/2$ around the y-axis. Write down the 2×2 unitary matrix U corresponding to the net rotation.

(b) The net rotation U from (a) is expressible as a single rotation. By what angle? Around what axis?

(c) To lowest nonzero order in ϵ, write down the 2×2 unitary transformation U_ϵ corresponding to a rotation of ϵ around the x-axis, followed by a rotation of ϵ around the y-axis, then a rotation of $-\epsilon$ around the x-axis, and finally a rotation of $-\epsilon$ around the y-axis.

(d) The net approximate rotation U_ϵ from (c) is expressible as a single rotation. By what angle? Around what axis?

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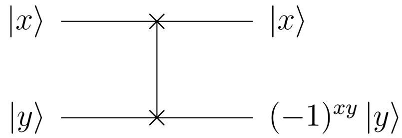
3) Cluster States

A *controlled phase gate* applies a phase of -1 to $|11\rangle$ and leaves all other logical states unchanged. That is,

$$|00\rangle \to |00\rangle, \quad |01\rangle \to |01\rangle, \quad |10\rangle \to |10\rangle, \quad |11\rangle \to -|11\rangle$$

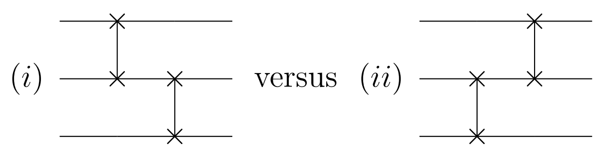
We depict controlled phase gates with the following diagram.



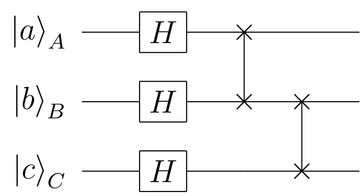
[Note that the phase factor $(-1)^{xy}$ can just as well be associated with the output ket $|x\rangle$. Controlled phase gates are symmetric.]

(a) Show how to construct a CNOT using controlled phase gates and single qubit rotations.

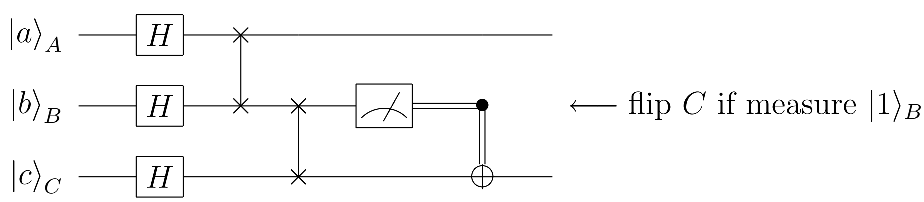
(b) How do the following two circuits differ in their effect?



(c) What state is produced by the following circuit?



(d) Consider extending the circuit in (c) by measuring qubit B in the $\{|0\rangle, |1\rangle\}$ basis and then flipping the qubit C if the result of the measurement is $|1\rangle_B$.



What is the joint state of qubits A and C after this procedure...

(i) ... if measurement of qubit B yields $|1\rangle_B$?

(ii) ... if you do not yet know the measurement result for qubit B?

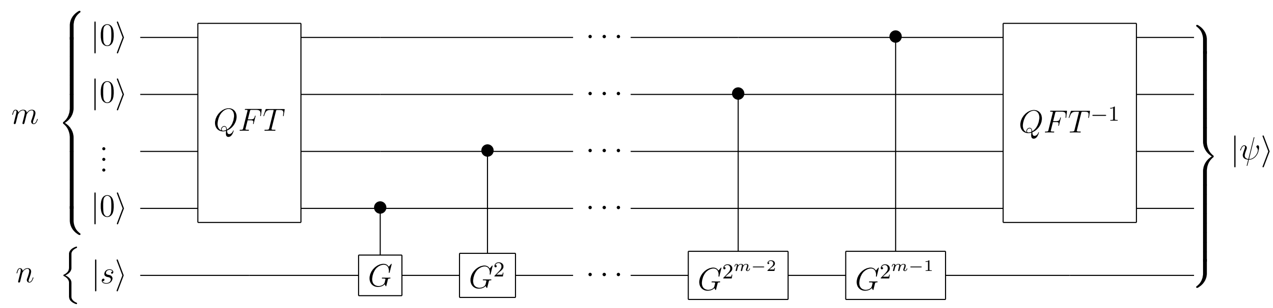
4) Quantum Phase Estimation for Grover's Algorithm

Let G denote the Grover step:

$$-\boxed{G}- \iff \big(\mathbb{I} - 2|s\rangle\langle s|\big)\big(2|w\rangle\langle w| - \mathbb{I}\big),$$

where $N = 2^n$, $|w\rangle$ is the target state, and $|s\rangle = \dfrac{1}{\sqrt{N}}\displaystyle\sum_{x=0}^{N-1}|x\rangle$.

(a) What is the output of the following circuit?



(b) Suppose $m = n$. What information do you get by measuring the first m bits of $|\psi\rangle$? After measurement, what state is registered by the second n bits of $|\psi\rangle$?

(c) How do your conclusions for part (b) change if $m \ll n$? If $m \gg n$?