2.111/18.435J Solutions 1.

PRINT IN LARGE LETTERS (Family Name) (First Name) (Middle Name) (Course) (Year)

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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(1) Buzz Words

(a) partial trace

(b) co-rotating frame

(c) Hope tensor

(d) Kraus operators effects

(e) quantum error-correcting code

(2)

$\Uparrow$ $B = 0.10$ Tesla $\qquad e^-$ $\downarrow$ ground state $\quad E_{\downarrow} = \mu \cdot B$

$= -9 \cdot 10^{-26}$ joule

$\uparrow$ excited state $\quad E_{\uparrow} = -\mu B$

$= +9 \cdot 10^{-24}$ joule

Larmor frequency $\Rightarrow$ $\hbar\omega = E_{\uparrow} - E_{\downarrow} = 1.8 \cdot 10^{-25}$ joule

$$\Rightarrow \omega \approx \frac{1.8 \cdot 10^{-25} \text{ joule}}{\hbar = 1.0546 \cdot 10^{-34} \text{ joule sec}} \approx 1.8 \cdot 10^{9} \frac{\text{radian}}{\text{sec}}$$

Hamiltonian $\qquad H = \dfrac{\hbar\omega}{2}\sigma_z \qquad (\downarrow$ = ground state$)$

time evolution $\qquad e^{-i\frac{\omega t \sigma_z}{2}}$ = $\underline{\text{right}}$ $\underline{\text{handed}}$ precession about z-axis

$\Rightarrow$ $\underline{\text{clockwise}}$ precession.

$\Rightarrow$ (2a) oscillating field should either be polarized (i) clockwise about z-axis or (ii) horizontally along the x-axis.

If the field is polarized horizontally along the x-axis, then it is a superposition of clockwise + counterclockwise rotating terms. The counterclockwise term can be ignored via the <u>co-rotating</u> <u>wave</u> <u>approximation</u>. (It gives rise to small, rapid <u>Bloch-Siegert</u> <u>oscillations</u>.)

(2b) In the co-rotating frame the Hamiltonian is

$$\frac{\hbar}{2}\left((\omega_0 - \omega)\sigma_z + \gamma \sigma_x \right) \qquad \gamma \text{ positive}$$

where $\left|\frac{\hbar\gamma}{2}\right| = |\mu B_{osc.}| \Rightarrow \gamma = \frac{2 \cdot 9 \cdot 10^{-24} \,J/T}{10^{-34}\, J s} \cdot 10^{-2}\, T = 1.8 \cdot 10^{8}\, \text{rad/sec}$

This induces a clockwise rotation about the $\begin{pmatrix} \gamma \\ 0 \\ \omega_0 - \omega \end{pmatrix}$ axis in the co-rotating frame. (un-normalized)

To effect a rotation about the $\frac{1}{\sqrt{2}}(\hat{x} - \hat{z})$ axis, we want $\gamma = -(\omega_0 - \omega)$

$$\Rightarrow \boxed{\omega = \omega_0 + \gamma = 1.98 \cdot 10^{9}\, \text{rad/sec}}$$

(2c) The frequency of the rotation in the co-rotating frame is

$$\Omega = \sqrt{(\omega_0 - \omega)^2 + \gamma^2} \simeq \sqrt{2}\,\gamma$$

we want $\Omega t = \pi \Rightarrow$

$$t = \frac{\pi}{\sqrt{(\omega_0-\omega)^2 + \gamma^2}} = \frac{\pi}{\sqrt{2}\,\gamma} = \frac{\pi}{\sqrt{2} \cdot 1.8 \cdot 10^{8}} \simeq 10^{-8}\, \text{sec.}$$

(2d) Our rotation in the co-rotating frame is

$$e^{-i\frac{\pi}{2}(\sigma_x - \sigma_z)/\sqrt{2}} = -i\,(\sigma_x - \sigma_z)/\sqrt{2} = -i \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Compared to a Hadamard, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. ~~The~~ Our rotation takes $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $-\frac{i}{\sqrt{2}}(-|0\rangle + |1\rangle)$ and $|1\rangle$ to $-i(|0\rangle + |1\rangle)$.

(8) Open quantum systems

$$H_{SE} = \frac{\hbar}{2}\left(\omega(a^\dagger a + \tfrac{1}{2}) + \hbar\omega_0 \hat{\sigma}_z + \gamma(a\sigma^+ + a^\dagger \sigma^-)\right)$$

Jaynes-Cummings Hamiltonian: oscillator = system
atom = environment

$\rho_{SE}(0) = \rho_S^{(0)} \otimes \rho_E(0)$ where $\rho_E(0) = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$.

Find a Lindbladian master equation for the oscillator on its own.

$$\rho_{SE}(0) \to \rho_{SE}(t) = e^{-iH_{SE}t} \rho_{SE}(0) e^{iH_{SE}t}$$

Expand to 2nd order in t:

$$\rho_{SE}(t) = \left(\mathbb{1} - iH_{SE}t - \frac{t^2}{2}H^2\right)\rho_{SE}(0)\left(\mathbb{1} + iH_{SE}t - \frac{t^2}{2}H^2\right) + O(t^3)$$

$$= \rho_{SE}(0) - i\left[H_{SE}, \rho_{SE}^{(0)}\right]t - \frac{1}{2}\left(H_{SE}^2\rho_{SE} - 2H_{SE}\rho_{SE}H_{SE} + \rho_{SE}H_{SE}^2\right)t^2 + O(t^3)$$

Now trace over the environment; using $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$

$$\rho_S(t) = \mathrm{tr}_E\, \rho_{SE}(0) - i\, \mathrm{tr}_E\left[H_{SE}, \rho_S^{(0)} \otimes \rho_E^{(0)}\right]t$$
$$- \frac{1}{2}\mathrm{tr}_E\left(H_{SE}^2\, \rho_S^{(0)} \otimes \rho_E^{(0)} - 2H_{SE}\rho_S^{(0)} \otimes \rho_E^{(0)} H_{SE} + \rho_S^{(0)} \otimes \rho_E^{(0)} H_{SE}^2\right)$$

This is what I was looking for:

$$\boxed{\rho_S(t) - \rho_S(0) = \mathrm{tr}_E\, e^{-iH_{SE}t}\rho_S(0) \otimes \rho_E(0) e^{iH_{SE}t} \quad \text{to second order in } t}$$

(3) Continued: The rest is just slogging away!

Let's do this term by term:

$tr_E\, \rho_s(0) \otimes \rho_E(0) = \rho_s(0)$

$-it\, tr_E\, [H_{SE}, \rho_s \otimes \rho_E]$

$= \frac{-it}{\hbar} tr\left(H_{SE}\, \rho_s \otimes \rho_E - \rho_s \otimes \rho_E\, H_{SE} \right)$

$H_{SE} = H_S + H_E + H_{int}$

$H_S = \frac{\hbar}{2}(\omega a^\dagger a + 1/2)$, $H_E = \frac{\hbar}{2}(\omega_0 \sigma_z)$, $H_{int} = \frac{\hbar}{2} \gamma (a\sigma^+ + a^\dagger \sigma^-)$

- The first term just gives $-it\, [H_s, \rho_s(0)]$, which is the normal time evolution.
- The second term gives **zero**: it commutes with $\rho_E(0)$.
- the third term also gives **zero** because $tr_E\, \rho_E(0)\sigma^+ = tr_E\, \rho_E \sigma^- = 0$.

So the first order term is just the ordinary system Hamiltonian

Now look at second order term: Here we have to look first at

$-\frac{t^2}{2} tr_E\, H^2 \rho = -\frac{t^2}{2} tr_E \left(H_s + H_E + H_{SE} \right)\left(H_s + H_E + H_{SE} \right) \rho_s \otimes \rho_E$

$+\frac{t^2}{2} tr_E\, H \rho H = +\frac{t^2}{2} tr_E \left((H_s + H_E + H_{SE})\, \rho_s \otimes \rho_E\, (H_s + H_E + H_{SE}) \right)$

$-\frac{t^2}{2} tr_E\, \rho H^2 = -\frac{t^2}{2} tr_E \left(\rho_s \otimes \rho_E\, (H_s + H_E + H_{SE})^2 \right)$

Note: • H_S and H_E commute $\Rightarrow$ $H_S H_E$ terms add up to zero, as do H_E^2 terms

• H_S^2 terms just give the ordinary 2nd order terms for H_S.

(8) Continued:

- $H_E \cdot H_{SE}$ terms all give zero once partial trace has been taken
- $H_S \cdot H_{SE}$ terms all give zero once partial trace has been taken

So the only terms left after taking the partial trace are the $-\frac{t^2}{2}\left(H_S^2\rho_S - 2H_S\rho_S H_S + H^2\rho_S\right) \Rightarrow$ normal Hamiltonian evolution.

Together with $-\frac{t^2}{2}\left(H_{SE}^2\rho_S - 2H_{SE}\rho H_{SE} + \rho_S H_{SE}^2\right)$

so let's do this latter set:

Now $H_{SE}^2 = \gamma^2(a\sigma^+ + a^\dagger\sigma^-)(a\sigma^+ + a^\dagger\sigma^-) = \gamma^2\left(aa^\dagger\sigma^+\sigma^- + a^\dagger a\,\sigma^-\sigma^+\right)$

where $\sigma^+\sigma^- = \frac{1}{2}(\sigma_x + i\sigma_y)\frac{1}{2}(\sigma_x - i\sigma_y)$
$= \frac{1}{4}\left(\sigma_x^2 + \sigma_y^2 - i\sigma_x\sigma_y + i\sigma_y\sigma_x\right) = \frac{1}{4}\left(\mathbb{1} + \mathbb{1} + 2\sigma_z\right) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

Similarly, $\sigma^-\sigma^+ = \begin{pmatrix}0&0\\0&1\end{pmatrix}$, so $H_{SE}^2 = \gamma^2\left(aa^\dagger(\mathbb{1}+\sigma_z)/2 + a^\dagger a(\mathbb{1}-\sigma_z)/2\right)$

But $[a, a^\dagger] = aa^\dagger - a^\dagger a = \mathbb{1}$, so this $\Rightarrow$

$$H_{SE}^2 = a^\dagger a \cdot \mathbb{1} + \underbrace{a^\dagger a \cdot (\mathbb{1}+\sigma_z)/2}_{=\,a^\dagger a\begin{pmatrix}1&0\\0&0\end{pmatrix}}$$

so $-\frac{t^2}{2}\gamma^2 \mathrm{tr}_E\Big(\cdot(a\sigma^+ + a^\dagger\sigma^-)(a\sigma^+ + a^\dagger\sigma^-)\rho_S\otimes\rho_E$
$-2(a\sigma^+ + a^\dagger\sigma^-)\rho_S\otimes\rho_E(a\sigma^+ + a^\dagger\sigma^-)$
$+\rho_S\otimes\rho_E(a\sigma^+ + a^\dagger\sigma^-)(a\sigma^+ + a^\dagger\sigma^-)\Big)$

(3 continued)

We obtain a master equation of the form:

$$\frac{\partial \rho_s}{\partial t} = -i\left[H_s, \rho_s\right] + \frac{\gamma_1}{2}\left(-a^\dagger a \rho_s + 2a^\dagger \rho_s a - \rho_s a^\dagger a\right)$$

$$+ \frac{\gamma_2}{2}\left(-aa^\dagger \rho_s + 2a\rho_s a^\dagger - \rho_s a a^\dagger\right)$$

no Lamb shift!

Where $\gamma_1 = \gamma_2/2$

The first term, with γ_1, excites the oscillator (+ takes atom from ~~ground~~ excited state to ~~excite~~ ground state)

The second term, with γ_1, de-excites the oscillator + takes atom from ground state to excited state.

(4b) For very short times: $(\gamma t)^2 \ll \omega t$
$\Rightarrow t \ll \frac{\omega}{\gamma^2}$, we can ignore the dissipative terms.

For ~~longer~~ intermediate times $(\gamma t)^2 \approx \omega t$, we should be OK
$t \approx \omega/\gamma^2$

For long times $t \gg \omega/\gamma^2$, coherence kicks in and our model is no good.

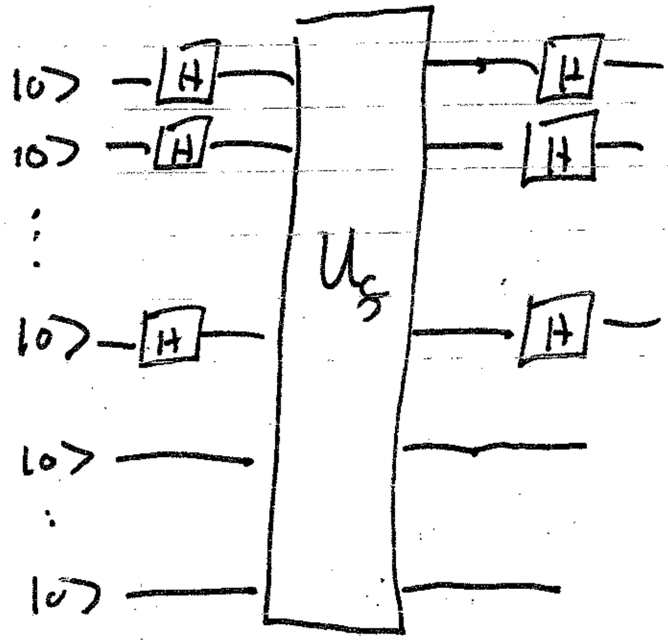
(4) Simon's Algorithm:

$f(x): \{0,1\}^n \to \{0,1\}^n$

promise $f(x) = f(x \oplus s)$ for some s (bitwise + mod 2)

$U_f |x, z\rangle = |x, z+s\rangle$ i.e. f 2-1

Circuit



measure $\Rightarrow$ get y such that $y \cdot s = 0$.

$\Rightarrow$ repeat $O(n)$ times to find s

Steps • $|00..0, 00..0\rangle \to 2^{-n/2} \sum_x |x, 0\rangle$

$$\to 2^{-n/2} \sum_x |x, f(x)\rangle$$

$$\to 2^{-n} \sum_{x,y} (-1)^{x \cdot y} |y, f(x)\rangle$$

$$= \frac{1}{2} 2^{-n} \sum_{x,y} \left((-1)^{x \cdot y} + (-1)^{(x \oplus s) \cdot y} \right) |y, f(x)\rangle$$

$$= \frac{1}{2} 2^{-n} \sum_{x,y} (-1)^{x \cdot y} \left(1 + (-1)^{y \cdot s} \right) |y, f(x)\rangle$$

$\Rightarrow$ first register contains y such that $y \cdot s = 0$!

(5) Quantum cryptography

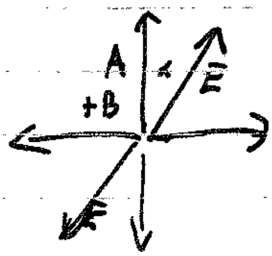
(5a) • Alice + Bob check half the qubits + discard the rest

• Of the qubits they check, half are measured by Eve in the correct basis → these she intercepts successfully

• Of the qubits that ~~the~~ Eve measures in the wrong basis, half are measured by Bob to be in the same state Alice sent, and half are measured to be in the wrong state

⇒ $\boxed{\tfrac{1}{8} \text{ of the time Alice and Bob detect Eve}}$

(5b) ⇒ $\boxed{\tfrac{1}{4} \text{ time Eve intercepts qubit without Alice + Bob knowing}}$

(5c) 

$\frac{1}{4}$ time A+B both use $\updownarrow$ basis

$\frac{1}{4}$ time they use $\leftrightarrow$ basis

$\frac{1}{2}$ time they don't use same basis ⇒ throw out

• When A+B use $\updownarrow$ basis, the probability that Eve gets the right answer + is not detected is $\underbrace{\cos^2\alpha}_{\text{right answer}} \cdot \underbrace{\cos^2\alpha}_{\text{not detected}}$

• Eve gets right answer + is detected $= \cos^2\alpha \sin^2\alpha$

• When A+B use $\leftrightarrow$ basis, the probability that Eve gets the ~~wrong~~ right answer + is not detected $= \sin^2\alpha \cdot \sin^2\alpha$

• right answer + is detected $= \sin^2\alpha + \cos^2\alpha$

(5) Quantum Cryptography continued

(5c) A+B send $\updownarrow$ $\Rightarrow$

p(Eve correct, undetected) $= \cos^2\alpha \cdot \cos^2\alpha$

p(Eve correct, detected) $= \cos^2\alpha \cdot \sin^2\alpha$

p(Eve incorrect, undetected) $= \sin^2\alpha \cdot \cos^2\alpha$

p(Eve incorrect, undetected) $= \sin^2\alpha \cdot \sin^2\alpha$

$\Rightarrow$ p(Eve undetected) $= \cos^2\alpha$

Similarly, when A+B send $\leftrightarrow$

p(Eve undetected) $= \sin^2\alpha$

So all in all, p(Eve undetected)

$$\boxed{= \tfrac{1}{4}\cdot\cos^2\alpha + \tfrac{1}{4}\sin^2\alpha = \tfrac{1}{4} \text{ Just as before}}$$

$\Rightarrow$ Eve's probability of being detected doesn't depend on α!