MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.111J/18.435J/ESD.79
Quantum Computation
Fall 2004

Problem Set 3
Due: Tuesday, October 5 (in class)

Problem 1. For a composite system A and B, define

$$\mathrm{I}_{AB}^2 = (\mathrm{I}_X^{AB})^2 + (\mathrm{I}_Y^{AB})^2 + (\mathrm{I}_Z^{AB})^2$$

where

$$\mathrm{I}_j^{AB} = \frac{1}{2}\sigma_j^A \otimes I_B + \frac{1}{2} I_A \otimes \sigma_j^B \text{ for } j \in \{X, Y, Z\}.$$

Evaluate

$$\left\langle \mathrm{I}_{AB}^2 \right\rangle = {}_{AB}\langle\psi|\mathrm{I}_{AB}^2|\psi\rangle_{AB}$$

for the following states:

i) $|\psi\rangle_{AB} = (|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$
ii) $|\psi\rangle_{AB} = (|01\rangle_{AB} + |10\rangle_{AB})/\sqrt{2}$
iii) $|\psi\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$
iv) $|\psi\rangle_{AB} = (|00\rangle_{AB} - |11\rangle_{AB})/\sqrt{2}$.

Problem 2. Recall

$$\begin{aligned}[\sigma_X, \sigma_Y] &\equiv \sigma_X\sigma_Y - \sigma_Y\sigma_X \\ &= 2i\sigma_Z.\end{aligned}$$

Find the following commutation relations:

$[\sigma_X^A \otimes \sigma_X^B, \sigma_Y^A \otimes \sigma_Y^B]$, $[\sigma_Y^A \otimes \sigma_Y^B, \sigma_Z^A \otimes \sigma_Z^B]$, and $[\sigma_Z^A \otimes \sigma_Z^B, \sigma_X^A \otimes \sigma_X^B]$.

Problem 3. For the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$, derive

$$\rho_A = tr_B(\rho_{AB})$$

where

$$\rho_{AB} = |\psi\rangle_{AB}\langle\psi|.$$

Problem 4. A C-NOT gate can be represented by the following unitary operator:

$$U_{CNOT} = |0\rangle_A\langle 0| \otimes I_B + |1\rangle_A\langle 1| \otimes \sigma_X^B.$$

Verify that $U_{CNOT} = U_{CNOT}^{\dagger}$ and $U_{CNOT}^{2} = I$.

Problem 5. Suppose your systems have 3-D vector spaces with $\{|0\rangle, |1\rangle, |2\rangle\}$ as the basis. For the operators

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \ \omega = e^{2\pi i/3} \ \ (R|j\rangle = e^{2\pi i j/3}|j\rangle)$$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad T|j\rangle = |(j+1) \bmod 3\rangle$$

a) Find the commutation relation among *R* and *T*, $[R,T]$.
b) Show how Alice and Bob can start with $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$ and use operators $R^a T^b$, for *a* and *b* integers, to send two classical trits (9 different messages) using one qutrit of communication.