

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation

Fall 2004

**Problem Set 5**

**Solutions**

**Problem 1.** For the state  $|\psi\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle_1 |g(x)\rangle_2$ , where  $g(x)$  is a 1-1 function, find the partial trace  $\rho_1 \equiv \text{tr}_2(|\psi\rangle\langle\psi|)$  and calculate  $\langle + | \rho_1 | + \rangle^{\otimes n}$ .

**Solution:**

$$|\psi\rangle\langle\psi| = \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{f(x)+f(y)} |x\rangle_1 \langle y| \otimes |g(x)\rangle_2 \langle g(y)|$$

Now, the fact that  $g(x)$  is a 1-1 function implies that for  $x \neq y$ , we have  $g(x) \neq g(y)$ , and therefore,

$$\langle g(x) | g(y) \rangle = \delta_{xy} = \text{tr}(|g(x)\rangle\langle g(y)|)$$

Using the above relation, we have

$$\begin{aligned} \rho_1 &\equiv \text{tr}_2(|\psi\rangle\langle\psi|) \\ &= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{f(x)+f(y)} |x\rangle_1 \langle y| \otimes \text{tr}(|g(x)\rangle_2 \langle g(y)|) \\ &= \frac{1}{2^n} \sum_{x=0}^{2^n-1} \sum_{y=0}^{2^n-1} (-1)^{f(x)+f(y)} |x\rangle_1 \langle y| \delta_{xy} \\ &= \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{f(x)+f(x)} |x\rangle_1 \langle x| \\ &= \frac{1}{2^n} \sum_{x=0}^{2^n-1} |x\rangle_1 \langle x| \\ &= I_1 / 2^n. \end{aligned}$$

Now, the probability of measuring  $|+\rangle^{\otimes n}$  is as follows

$$\begin{aligned} \text{tr}(\rho_1 |+\rangle^{\otimes n} \langle +|) &= \langle + | \rho_1 | + \rangle^{\otimes n} \\ &= \frac{1}{2^n} \langle + | I_1 | + \rangle^{\otimes n} \\ &= \frac{1}{2^n}. \end{aligned}$$

**Problem 2.** Find  $H^{\otimes n} R_\alpha H^{\otimes n}$  and  $H^{\otimes n} T_\alpha H^{\otimes n}$  in simpler terms, where

$$R_\alpha = \sum_{x=0}^{2^n-1} (-1)^{x \cdot \alpha} |x\rangle \langle x|$$

and

$$T_\alpha = \sum_{x=0}^{2^n-1} |x \oplus \alpha\rangle \langle x|.$$

**Solution:**

We know

$$H^{\otimes n} = \frac{1}{2^{n/2}} \sum_{a,b=0}^{2^n-1} (-1)^{a \cdot b} |b\rangle \langle a|$$

$\Rightarrow$

$$\begin{aligned} H^{\otimes n} T_\alpha H^{\otimes n} &= \frac{1}{2^n} \sum_{a,b=0}^{2^n-1} \sum_{c,d=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{a \cdot b + c \cdot d} |c\rangle \langle d| x \oplus \alpha \langle x| b \rangle \langle a| \\ &= \frac{1}{2^n} \sum_{a,b=0}^{2^n-1} \sum_{c,d=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{a \cdot b + c \cdot d} |c\rangle \delta_{d, x \oplus \alpha} \delta_{x b} \langle a| \\ &= \frac{1}{2^n} \sum_{a=0}^{2^n-1} \sum_{c=0}^{2^n-1} \sum_{x=0}^{2^n-1} (-1)^{a \cdot x + c \cdot (x \oplus \alpha)} |c\rangle \langle a| \\ &= \frac{1}{2^n} \sum_{a=0}^{2^n-1} \sum_{c=0}^{2^n-1} (-1)^{c \cdot \alpha} |c\rangle \langle a| \sum_{x=0}^{2^n-1} (-1)^{a \cdot x + c \cdot x} \\ &= \frac{1}{2^n} \sum_{a=0}^{2^n-1} \sum_{c=0}^{2^n-1} (-1)^{c \cdot \alpha} |c\rangle \langle a| 2^n \delta_{ac} \\ &= \sum_{a=0}^{2^n-1} (-1)^{a \cdot \alpha} |a\rangle \langle a| \\ &= R_\alpha. \end{aligned}$$

Now, using  $H^{\otimes n} = (H^{\otimes n})^{-1}$ , we have

$$H^{\otimes n} R_\alpha H^{\otimes n} = T_\alpha.$$

**Hint:**  $\sum_{x=0}^{2^n-1} (-1)^{a \cdot x} = 2^n \delta(a)$ .

**Problem 3.** Find  $U_p R_p U_p^\dagger$  and  $U_p T_p U_p^\dagger$  in simpler terms, where

$$R_p = \sum_{x=0}^{p-1} \exp(2\pi x i / p) |x\rangle \langle x|$$

$$T_p = \sum_{x=0}^{p-1} |x + 1 \bmod p\rangle \langle x|$$

$$U_p = \frac{1}{\sqrt{p}} \sum_{x=0}^{p-1} \sum_{y=0}^{p-1} \exp(2\pi i xy / p) |y\rangle \langle x|$$

and  $p$  is a prime number.

**Solution:**

$$\begin{aligned} U_p T_\alpha U_p^\dagger &= \frac{1}{p} \sum_{a,b=0}^{p-1} \sum_{c,d=0}^{p-1} \sum_{x=0}^{p-1} \exp(2\pi i(cd - ab) / p) |c\rangle \langle d| x + 1 \rangle \langle x| b \rangle \langle a|, \text{ addition mod } p \\ &= \frac{1}{p} \sum_{a,b=0}^{p-1} \sum_{c,d=0}^{p-1} \sum_{x=0}^{p-1} \exp(2\pi i(cd - ab) / p) |c\rangle \delta_{d,x+1} \delta_{xb} \langle a| \\ &= \frac{1}{p} \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \sum_{x=0}^{p-1} \exp(2\pi i(cx + c - ax) / p) |c\rangle \langle a| \\ &= \frac{1}{p} \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \exp(2\pi ic / p) |c\rangle \langle a| \sum_{x=0}^{p-1} \exp(2\pi i(cx - ax) / p) \\ &= \frac{1}{p} \sum_{a=0}^{p-1} \sum_{c=0}^{p-1} \exp(2\pi ic / p) |c\rangle \langle a| p \delta_{ac} \\ &= \sum_{a=0}^{p-1} \exp(2\pi ia / p) |a\rangle \langle a| \\ &= R_p. \end{aligned}$$

Similarly, you can show that

$$U_p^\dagger R_p U_p = T_p$$

and

$$U_p R_p U_p^\dagger = T_p^\dagger.$$

**Hint:** 
$$\sum_{x=0}^{p-1} \exp(2\pi i ax / p) = \begin{cases} p & a \equiv 0 \pmod{p} \\ \frac{\exp(2\pi ia) - 1}{\exp(2\pi ia / p) - 1} = 0 & \text{otherwise} \end{cases}, \text{ for } a \in \mathbb{Z}.$$

**Problem 4.** Show that  $U_2 \otimes U_3 = P U_6 P^{-1}$  where  $U_p$  is defined in Problem 3, and  $P$  is a permutation matrix (a matrix with only one nonzero element 1 in each row and column).

**Solution:** It can be seen that

$$U_2 \otimes U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} U_3 & U_3 \\ U_3 & -U_3 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & 1 & 1 & \omega^3 & \omega^3 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^3 & \omega^5 & \omega \\ 1 & \omega^4 & \omega^2 & \omega^3 & \omega & \omega^5 \end{bmatrix}$$

and

$$U_6 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

where  $\omega = \exp(2\pi i/6)$ .

There is no such  $P$  that satisfies  $U_2 \otimes U_3 = PU_6P^{-1}$ . You can however find  $P_1$  and  $P_2$  such that  $U_2 \otimes U_3 = P_1U_6P_2$ . For instance,

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also, you can verify that  $U_2 \otimes U_3^\dagger = P_1U_6P_1^{-1}$ .