

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quantum Computation

Fall 2004

Problem Set 9

Solutions

Problem 1. For the position operator x and the momentum operator p (corresponding to $-i\hbar \frac{\partial}{\partial x}$), show that $[x, p] = i\hbar$.

Solution:

For any function $f(x)$, we have

$$\begin{aligned}[x, p]f(x) &= xpf(x) - px f(x) \\ &= x \frac{\partial}{\partial x} f(x) - \frac{\partial}{\partial x} (xf(x)) \\ &= -i\hbar x \frac{\partial}{\partial x} f(x) + i\hbar x \frac{\partial}{\partial x} (f(x)) + i\hbar f(x) \\ &= i\hbar f(x)\end{aligned}$$

\Rightarrow

$$[x, p] = i\hbar.$$

Problem 2. Verify the following equalities:

(a) $[a, a^\dagger] = 1$

(b) $[a, a^\dagger a] = a$

(c) $[a^\dagger, a^\dagger a] = -a^\dagger$

(d) $H = \hbar\omega(a^\dagger a + 1/2)$

where

$$a = \frac{1}{\sqrt{2\hbar\omega}}(\omega x + ip)$$

and

$$H = \frac{1}{2}(p^2 + \omega^2 x^2).$$

Solution:

$$\begin{aligned}\text{(a) } [a, a^\dagger] &= \frac{1}{2\hbar\omega}[\omega x + ip, \omega x - ip] \\ &= \frac{1}{2\hbar\omega}(\omega^2[x, x] + [p, p] - i\omega[x, p] + i\omega[p, x])\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\hbar\omega}(-i\omega(i\hbar) + i\omega(-i\hbar)) \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad [a, a^\dagger a] &= aa^\dagger a - a^\dagger aa \\
&= (aa^\dagger - a^\dagger a)a \\
&= [a, a^\dagger]a \\
&= a.
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad [a^\dagger, a^\dagger a] &= a^\dagger(a^\dagger a - aa^\dagger) \\
&= -a^\dagger.
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad \hbar\omega(a^\dagger a + 1/2) &= \hbar\omega\left[\frac{1}{2\hbar\omega}(\omega x - ip)(\omega x + ip) + \frac{1}{2}\right] \\
&= (\omega^2 x^2 + p^2 + i\omega[x, p] + \hbar\omega)/2 \\
&= \frac{1}{2}(p^2 + \omega^2 x^2) \\
&= H.
\end{aligned}$$

Problem 3. For $\hbar = \omega = 1$, verify $a|0\rangle = 0$, where

$$|0\rangle = \int e^{-x^2/2} |x\rangle dx.$$

Solution:

$$\begin{aligned}
a|0\rangle &= \frac{1}{\sqrt{2\hbar\omega}}(\omega x + ip) \int e^{-x^2/2} |x\rangle dx \\
&= \frac{1}{\sqrt{2\hbar\omega}} \int \left(\omega x e^{-x^2/2} + i(-i\hbar) \frac{\partial}{\partial x} e^{-x^2/2} \right) |x\rangle dx \\
&= \frac{1}{\sqrt{2\hbar\omega}} \int (\omega - \hbar) x e^{-x^2/2} |x\rangle dx \\
&= 0
\end{aligned}$$

Problem 4. Verify

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

and

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

Also, show that $|n\rangle$ is an eigenstate of H with energy (eigenvalue) $\hbar\omega(n + 1/2)$.

Solution:

$$\begin{aligned}
a^\dagger |n\rangle &= a^\dagger \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \\
&= \sqrt{n+1} \frac{(a^\dagger)^{n+1}}{\sqrt{(n+1)!}} |0\rangle \\
&= \sqrt{n+1} |n+1\rangle.
\end{aligned}$$

$$\begin{aligned}
a |n\rangle &= a \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \\
&= a a^\dagger \frac{(a^\dagger)^{n-1}}{\sqrt{n!}} |0\rangle \\
&= (1 + a^\dagger a) \frac{1}{\sqrt{n}} \frac{(a^\dagger)^{n-1}}{\sqrt{(n-1)!}} |0\rangle \\
&= (|n-1\rangle + a^\dagger a |n-1\rangle) / \sqrt{n}
\end{aligned}$$

but, by induction, we have

$$\begin{aligned}
a^\dagger a |n-1\rangle &= \sqrt{n-1} a^\dagger |n-2\rangle = (n-1) |n-1\rangle \\
\Rightarrow \\
a |n\rangle &= \sqrt{n} |n-1\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
H |n\rangle &= \hbar\omega(a^\dagger a + 1/2) |n\rangle \\
&= \hbar\omega(n + 1/2) |n\rangle.
\end{aligned}$$

Problem 5. Show that $[C, D] = [C, H] = [D, H] = 0$, where

$$\begin{aligned}
C &= \omega(a^\dagger a + \sigma_z / 2), \\
D &= \kappa(a^\dagger \sigma_- + a \sigma_+) - (\Delta\omega / 2) \sigma_z,
\end{aligned}$$

and

$$H = \hbar(C + D + \omega/2).$$

(Remember that $\sigma_\pm = (\sigma_x \pm i\sigma_y) / 2$, or equivalently $\sigma_+ = |0\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 0|$.)

Solution:

We have

$$\begin{aligned}
[a^\dagger a, \sigma_z] &= 0 \\
\sigma_z \sigma_+ &= (|0\rangle\langle 0| - |1\rangle\langle 1|) |0\rangle\langle 1| = |0\rangle\langle 1| = \sigma_+ \\
\sigma_+ \sigma_z &= |0\rangle\langle 1| (|0\rangle\langle 0| - |1\rangle\langle 1|) = -|0\rangle\langle 1| = -\sigma_+
\end{aligned}$$

$$\begin{aligned}\sigma_- \sigma_Z &= |1\rangle\langle 0|(|0\rangle\langle 0| - |1\rangle\langle 1|) = |1\rangle\langle 0| = \sigma_- \\ \sigma_Z \sigma_- &= (|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle\langle 0| = -|1\rangle\langle 0| = -\sigma_-\end{aligned}$$

\Rightarrow

$$\begin{aligned}[C, D] &= [\omega(a^\dagger a + \sigma_Z/2), \kappa(a^\dagger \sigma_- + a \sigma_+) - (\Delta\omega/2)\sigma_Z] \\ &= \omega\kappa[a^\dagger a, a^\dagger] \otimes \sigma_- + \omega\kappa[a^\dagger a, a] \otimes \sigma_+ + \frac{\omega\kappa}{2} a^\dagger \otimes [\sigma_Z, \sigma_-] + \frac{\omega\kappa}{2} a \otimes [\sigma_Z, \sigma_+] \\ &= \omega\kappa a^\dagger \otimes \sigma_- - \omega\kappa a \otimes \sigma_+ - \omega\kappa a^\dagger \otimes \sigma_- + \omega\kappa a \otimes \sigma_+ \\ &= 0.\end{aligned}$$

$$[C, H] = [C, \hbar(C + D + \omega/2)] = \hbar([C, C] + [C, D] + [C, \omega I/2]) = 0.$$

$$[D, H] = [D, \hbar(C + D + \omega/2)] = \hbar([D, C] + [D, D] + [D, \omega I/2]) = 0.$$

Problem 6. Find the cluster state for a square. In other words, find the state $|\psi\rangle$ such that

$$\sigma_X^A \sigma_Z^B \sigma_Z^D |\psi\rangle = \sigma_X^B \sigma_Z^C \sigma_Z^A |\psi\rangle = \sigma_X^C \sigma_Z^D \sigma_Z^B |\psi\rangle = \sigma_X^D \sigma_Z^A \sigma_Z^C |\psi\rangle = |\psi\rangle$$

where the super indices A, B, C, and D are associated with the vertices of the square.

Solution:

A general eigenstate $|\psi\rangle$ with eigenvalue 1 for $\sigma_X^A \sigma_Z^B \sigma_Z^D$ is as follows:

$$\begin{aligned}|\psi\rangle &= \alpha_1 |+\rangle_A |0\rangle_B |0\rangle_C |0\rangle_D + \alpha_2 |+\rangle_A |0\rangle_B |1\rangle_C |0\rangle_D + \\ &\alpha_3 |+\rangle_A |1\rangle_B |0\rangle_C |1\rangle_D + \alpha_4 |+\rangle_A |1\rangle_B |1\rangle_C |1\rangle_D + \\ &\alpha_5 |-\rangle_A |0\rangle_B |0\rangle_C |1\rangle_D + \alpha_6 |-\rangle_A |0\rangle_B |1\rangle_C |1\rangle_D + \\ &\alpha_7 |-\rangle_A |1\rangle_B |0\rangle_C |0\rangle_D + \alpha_8 |-\rangle_A |1\rangle_B |1\rangle_C |0\rangle_D\end{aligned}$$

where $\sum_{k=1}^8 |\alpha_k|^2 = 1$. In order to satisfy the second equation stated in the problem, we have

$$\begin{aligned}\sigma_Z^A \sigma_X^B \sigma_Z^C |\psi\rangle &= \alpha_1 |-\rangle_A |1\rangle_B |0\rangle_C |0\rangle_D - \alpha_2 |-\rangle_A |1\rangle_B |1\rangle_C |0\rangle_D + \\ &\alpha_3 |-\rangle_A |0\rangle_B |0\rangle_C |1\rangle_D - \alpha_4 |-\rangle_A |0\rangle_B |1\rangle_C |1\rangle_D + \\ &\alpha_5 |+\rangle_A |1\rangle_B |0\rangle_C |1\rangle_D - \alpha_6 |+\rangle_A |1\rangle_B |1\rangle_C |1\rangle_D + \\ &\alpha_7 |+\rangle_A |0\rangle_B |0\rangle_C |0\rangle_D - \alpha_8 |+\rangle_A |0\rangle_B |1\rangle_C |0\rangle_D \\ &= |\psi\rangle\end{aligned}$$

\Rightarrow

$$\alpha_1 = \alpha_7, \alpha_3 = \alpha_5, \alpha_2 = -\alpha_8, \alpha_4 = -\alpha_6.$$

Similarly, by applying the other two operators we get:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = -\alpha_6 = \alpha_7 = -\alpha_8,$$

which after normalization results in the following cluster state:

$$\begin{aligned} |\psi\rangle = & \frac{1}{2} |+\rangle_A |0\rangle_B |+\rangle_C |0\rangle_D + \frac{1}{2} |+\rangle_A |1\rangle_B |+\rangle_C |1\rangle_D \\ & + \frac{1}{2} |-\rangle_A |0\rangle_B |-\rangle_C |1\rangle_D + \frac{1}{2} |-\rangle_A |1\rangle_B |-\rangle_C |0\rangle_D. \end{aligned}$$