PS1 Solutions

1 (exercise 2.1)

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence, the vectors are linearly dependent.

2 (exercise 2.6)

$$\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) = \left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right)^* = \sum_i \lambda_i{}^* (|v\rangle, |w_i\rangle)^* = \sum_i \lambda_i{}^* \left(|w_i\rangle, |v\rangle\right)$$

3 (exercise 2.7)

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

, so the vectors are othogonal. The normalized vectors are:

$$|w^{'}\rangle = \frac{1}{\sqrt{2}}|w\rangle, \quad |v^{'}\rangle = \frac{1}{\sqrt{2}}|v\rangle$$

4 (exercise 2.9)

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|; \quad \sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|; \quad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

5 (exercise 2.11)

σ_x:
eigenvectors corresponding to eigenvalues 1 and -1:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

diagonal representation:

$$\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

σ_y:
eigenvectors corresponding to eigenvalues 1 and -1:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}; \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

diagonal representation:

$$\frac{1}{2}\begin{bmatrix} 1 \\ i \end{bmatrix} \cdot \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix} \cdot \begin{bmatrix} 1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

σ_z:
eigenvectors corresponding to eigenvalues 1 and -1:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

diagonal representation:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

6 (exercise 2.13)

$$(|w\rangle\langle v|)^\dagger = ((w \cdot (v^*)^T)^*)^T = (((v^*)^T)^*)^T \cdot (w^*)^T = v \cdot (w^*)^T = |v\rangle\langle w|$$

7 (exercise 2.16)

$$P^2 = \sum |i\rangle\langle i| \sum |j\rangle\langle j| = \sum_{i=j} |i\rangle\langle j| = \sum |i\rangle\langle i| = P$$

8 (exercise 2.17)

1. if A is normal and hermitian then it has real eigenvalues
let $|\psi\rangle$ be an eigenvector with eigenvalue a.

$$\langle\psi|A|\psi\rangle = a$$

$$a^* = (\langle\psi|A|\psi\rangle)^\dagger = (\langle\psi|A^\dagger|\psi\rangle) = \langle\psi|A|\psi\rangle = a$$

Hence a is real.

1. if A is normal and has real eigenvalues, then it is hermitian
A is normal, so it can be written as $\sum \lambda_i |i\rangle\langle i|$, where the $|i\rangle$'s are eigenvectors with real eigenvalues λ_i and form an orthonormal basis.

$$A^\dagger = (\sum \lambda_i |i\rangle\langle i|)^\dagger = \sum \lambda_i^* |i\rangle\langle i| = \sum \lambda_i |i\rangle\langle i| = A$$

Hence A is hermitian.

9 (exercise 2.18)

Let $|\psi\rangle$ be an eigenvector of U with eigenvalue u.

$$U^{-1}U|\psi\rangle = |\psi\rangle = U^{-1}u|\psi\rangle$$

$$U^{-1}|\psi\rangle = \frac{1}{u}|\psi\rangle$$

$$\langle\psi|U|\psi\rangle = u$$

$$u^* = (\langle\psi|U|\psi\rangle)^\dagger = \langle\psi|U^\dagger|\psi\rangle = \langle\psi|U^{-1}|\psi\rangle = \frac{1}{u}$$

Hence $uu^* = 1$.

10 (exercise 2.22)

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two eigenvectors of A with different (real) eigenvalues a_1 and a_2.

$$\langle\psi_1|A|\psi_2\rangle = a_2\langle\psi_1|\psi_2\rangle$$

$$a_2\langle\psi_2|\psi_1\rangle = (a_2\langle\psi_1|\psi_2\rangle)^\dagger = (\langle\psi_1|A|\psi_2\rangle)^\dagger = \langle\psi_2|A^\dagger|\psi_1\rangle = \langle\psi_2|A|\psi_1\rangle = a_1\langle\psi_2|\psi_1\rangle$$

Hence, $\langle\psi_2|\psi_1\rangle = 0$ and the two eigenvectors are orthogonal.

11 (exercise 2.23)

Let $|\psi\rangle$ be an eigenvector of P with eigenvalue p.

$$p|\psi\rangle = P|\psi\rangle = P^2|\psi\rangle = Pp|\psi\rangle = p^2|\psi\rangle$$

Hence $p = p^2$ or $p = 0, 1$

12 (exercise 2.24)

$$A = \frac{A + A^\dagger}{2} + \frac{A - A^\dagger}{2} = \frac{A + A^\dagger}{2} + i\frac{A - A^\dagger}{2i} = B + iC$$

You can easily verify that B and C are Hermitian.
Fact: $\langle\psi|H|\psi\rangle$ is real for any hermitian matrix H and vector $|\psi\rangle$. (write $|\psi\rangle$ as a linear combination of eigenvectors; then $\langle\psi|H|\psi\rangle$ is a linear combination with positive coefficients of the corresponding eigenvalues)

$$\langle\psi|A|\psi\rangle = \langle\psi|B|\psi\rangle + i\langle\psi|C|\psi\rangle$$

First quantity is positive, hence real. However, the sum on the right is real iff C is zero. Hence $A = B$ hermitian.