PS2 Solutions

1 (exercise 2.27)

$$X \otimes Z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; I \otimes X == \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; X \otimes I == \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};$$

not commutative

2 (exercise 2.28)

Complex conjugation and transpose equalities are easily verified from the matrix representation of the tensor product. For the adjoint:

$$(A \otimes B)^\dagger = ((A \otimes B)^*)^T = (A^*)^T \otimes (B^*)^T = A^\dagger \otimes B^\dagger$$

3 (exercise 2.29)

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) = (U_1^\dagger U_1 \otimes U_2^\dagger U_2) = I \otimes I = I$$

Hence, $U_1 \otimes U_2$ is unitary.

4 (exercise 2.30)

$$(H_1 \otimes H_2)^\dagger = H_1^\dagger \otimes H_2^\dagger = H_1 \otimes H_2$$

Hence, $H_1 \otimes H_2$ is hermitian.

5 (exercise 2.32)

$$P_1 \otimes P_2 = \sum |i\rangle\langle i| \otimes \sum |j\rangle\langle j| = \sum |i\rangle \otimes |j\rangle\langle i| \otimes \langle j|$$

Hence, $P_1 \otimes P_2$ is a projection.

6 (exercise 2.33)

Use induction: formula true for $n = 1$ suppose it's true for n and prove it for $n + 1$.

$$H^{\otimes n+1} = H^{\otimes n} \otimes H = \frac{1}{\sqrt{2^n}} \sum_{x,y\in\{0,1\}^n} (-1)^{x\cdot y}|x\rangle\langle y| \otimes \frac{1}{\sqrt{2}} \sum_{x',y'\in\{0,1\}} (-1)^{x'\cdot y'}|x'\rangle\langle y'|$$

$$H^{\otimes n+1} = \frac{1}{\sqrt{2^{n+1}}} \sum_{\substack{x,y\in\{0,1\}^n \\ x',y'\in\{0,1\}}} (-1)^{x\cdot y + x'\cdot y'}|xx'\rangle\langle yy'| = \frac{1}{\sqrt{2^{n+1}}} \sum_{x,y\in\{0,1\}^{n+1}} (-1)^{x\cdot y}|x\rangle\langle y|$$

For the second part:

$$H^{\otimes 2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

7 (exercise 2.37)

Easy, just do the algebra :)

8 (exercise 2.38)

Same as above.

9 (exercise 2.47)

$(i[A,B])^\dagger = -i(AB - BA)^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = -i(BA - AB) = i(AB_B A) = i[A,B]$
Hence, $i[A,B]$ is hermitian.

10 (exercise 2.51)

$H^\dagger = H$; Check that $H^2 = I$

11 (exercise 2.52)

See above.

12 (exercise 2.53)

Eigenvalues are +1 and -1; the corresponding eigenvectors are:

$$\frac{1}{\sqrt{4 \mp 2\sqrt{2}}}\begin{bmatrix} 1 \\ \pm\sqrt{2} - 1 \end{bmatrix}$$

13 extra problem

$$e^{-i\theta A}e^{-i\theta B}e^{i\theta A}e^{i\theta B} \approx (1 - i\theta A - \frac{\theta^2 A^2}{2})(1 - i\theta B - \frac{\theta^2 B^2}{2})(1 + i\theta A - \frac{\theta^2 A^2}{2})(1 + i\theta B - \frac{\theta^2 B^2}{2})$$

$$\approx 1 - i\theta A - i\theta B + i\theta A + i\theta B - \frac{\theta^2 A^2}{2} - \frac{\theta^2 B^2}{2} - \frac{\theta^2 A^2}{2} - \frac{\theta^2 B^2}{2} -$$

$$-\theta^2 AB + \theta^2 A^2 + \theta^2 AB + \theta^2 BA + \theta^2 B^2 - \theta^2 AB$$

$$e^{-i\theta A}e^{-i\theta B}e^{i\theta A}e^{i\theta B} \approx 1 - \theta^2 [A, B] + O(\theta^3)$$