MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.111J/18.435J/ESD.79
Quantum Computation
Fall 2004

MIDTERM EXAM
Thursday, October 28

Problem 1. In NMR quantum computing, a Hadamard gate is implemented by rotating around the axis $(\vec{x}+\vec{z})/\sqrt{2}$. Compute the matrix obtained by rotation around this axis by π radians, and compare to a Hadamard gate.

Solution:

If we denote the rotation by angle θ about $(\vec{x}+\vec{z})/\sqrt{2}$ by $R(\theta)$, we have

$$\begin{aligned} R(\theta) &= \exp[-i(\theta/2)(\sigma_X+\sigma_Z)/\sqrt{2}\} \\ &= \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(\sigma_X+\sigma_Z)/\sqrt{2} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} R(\pi) &= -i(\sigma_X+\sigma_Z)/\sqrt{2} \\ &= \frac{-i}{\sqrt{2}}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) \\ &= \frac{-i}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \\ &= -iH \end{aligned}$$

where H is the Hadamard gate.

Problem 2. Let

$$H = \frac{1}{2}(\sigma_X\otimes\sigma_X+\sigma_Y\otimes\sigma_Y+\sigma_Z\otimes\sigma_Z+I\otimes I)$$

be an operator on two qubits.

a) Find H^2 and write it in a simple form.
b) Using (a), find $\exp(-i\pi H/4)$ and $\exp(-i\pi H/2)$.
c) Find the eigenvalues of H.
d) Find a set of orthonormal eigenstates of H.

Solution:

a) We have

$$H^2 = \frac{1}{2}(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I)H\,.$$

Note that

$$\begin{aligned}\frac{1}{2}(\sigma_X \otimes \sigma_X)H &= \frac{1}{4}(\sigma_X \otimes \sigma_X)(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I) \\ &= \frac{1}{4}(\sigma_X\sigma_X \otimes \sigma_X\sigma_X + \sigma_X\sigma_Y \otimes \sigma_X\sigma_Y + \sigma_X\sigma_Z \otimes \sigma_X\sigma_Z + \sigma_X \otimes \sigma_X) \\ &= \frac{1}{4}(I \otimes I + i\sigma_Z \otimes i\sigma_Z + (-i)\sigma_Y \otimes (-i)\sigma_Y + \sigma_X \otimes \sigma_X) \\ &= \frac{1}{4}(I \otimes I - \sigma_Z \otimes \sigma_Z - \sigma_Y \otimes \sigma_Y + \sigma_X \otimes \sigma_X)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{2}(\sigma_Y \otimes \sigma_Y)H &= \frac{1}{4}(-\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y - \sigma_Z \otimes \sigma_Z + I \otimes I) \\ \frac{1}{2}(\sigma_Z \otimes \sigma_Z)H &= \frac{1}{4}(-\sigma_X \otimes \sigma_X - \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z + I \otimes I) \\ \frac{1}{2}(I \otimes I)H &= \frac{H}{2}\,.\end{aligned}$$

Adding up these four relations, one can obtain

$$H^2 = I \otimes I\,.$$

b) Using equation (4.7) of N&C, we have

$$\exp(i\theta H) = \cos(\theta) I \otimes I + i\sin(\theta) H$$

$\Rightarrow$

$$\exp(-i\pi H/4) = \sqrt{2} I \otimes I/2 - i\sqrt{2} H/2$$

and

$$\exp(-i\pi H/2) = -iH\,.$$

c) Using Problem 1(b) in Problem Set 2, it can be seen that the only possible values for the eigenvalues are +1 and –1.

d) You can easily verify that the Bell states, described in the first problem of Problem Set 3, are one possible set of eigenstates. (In fact, $H = \mathrm{I}^2_{AB} - I \otimes I$.) The first state in that problem, the singlet sate, has eigenvalue –1 and the other three have eigenvalues +1.

Problem 3. Let N be an integer larger than 5. Consider the following state:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x \bmod N\rangle_A \otimes |3x \bmod N\rangle_B \otimes |5x \bmod N\rangle_C \ .$$

Find the output state if we take a quantum Fourier transform modulus N on each of the registers *A*, *B*, and *C*. That is, if we denote the corresponding QFT operators to each system by U_A, U_B, and U_C, find $U_A \otimes U_B \otimes U_C |\psi\rangle$. Write your answer in the basis $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}^{\otimes 3}$, and show that it is the superposition of equally probable states. What is this probability?

Solution:

$$\begin{aligned}
U_A \otimes U_B \otimes U_C |\psi\rangle &= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} U_A |x \bmod N\rangle_A \otimes U_B |3x \bmod N\rangle_B \otimes U_C |5x \bmod N\rangle_C \\
&= \left(\frac{1}{\sqrt{N}}\right)^4 \sum_{x=0}^{N-1} \sum_{k=0}^{N-1} e^{2\pi i x k} |k\rangle_A \otimes \sum_{m=0}^{N-1} e^{2\pi i (3x) m} |m\rangle_B \otimes \sum_{n=0}^{N-1} e^{2\pi i (5x) n} |n\rangle_C \\
&= \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} e^{2\pi i (k+3m+5n) x} |k\rangle_A |m\rangle_B |n\rangle_C \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |k\rangle_A |m\rangle_B |n\rangle_C \sum_{x=0}^{N-1} e^{2\pi i (k+3m+5n) x} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |k\rangle_A |m\rangle_B |n\rangle_C \, N \delta_{k, -3m-5n \bmod N} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |-3m-5n \bmod N\rangle_A |m\rangle_B |n\rangle_C \ .
\end{aligned}$$

This is the superposition of N^2 states each with probability of occurrence $1/N^2$.