Problem 1

Find the phase factor $\phi$ in the following diagram

$$|0\rangle \xrightarrow{\mathbf{e}^{i\frac{\pi}{4}\sigma_y}} \frac{\mathbf{e}^{i\phi}}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Problem 2

Show how to construct CCNOT from CNOT and single-qubit rotations (up to an overall phase). Suggestion: Use google to search for a solution, but keep a list of what websites were used.

Problem 3

Assume that a singlet state is shared between Alice and Bob. Verify that if we measure $|0\rangle, |1\rangle$ on each qubit separately, we obtain the following probabilities of outcomes:

$$P(01) = P(10) = \frac{\langle \psi | (|0\rangle_A \langle 0| \otimes |1\rangle_B \langle 1|) |\psi\rangle}{2} = \frac{1}{2}$$

$$P(11) = 0$$

Problem 4

Verify this:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = +1 |\rightarrow\rangle\langle \rightarrow| - 1 |\leftarrow\rangle\langle \leftarrow|$$

Problem 5

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B)$$

Rewrite $|\psi\rangle$ in terms of $|\rightarrow\rangle_A |\rightarrow\rangle_B, |\leftarrow\rangle_A |\leftarrow\rangle_B, |\rightarrow\rangle_A |\leftarrow\rangle_B, |\leftarrow\rangle_A |\rightarrow\rangle_B,$ and $|\leftarrow\rangle_A |\leftarrow\rangle_B,$ where for e.g. $|\uparrow\rangle = \frac{1}{\sqrt{2}} (|\rightarrow\rangle - |\leftarrow\rangle)$ etc.
Problem 6

We find that

$$\text{tr}_B |\psi\rangle_{AB} \langle \psi| = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, where $|\psi\rangle_{AB}$ has been defined in Problem 5. This tells us that state of A looks completely random, i.e. there is a 50% chance of it being up and 50% chance of it being down. In class we have calculated this in the $\sigma_z$ eigenstate basis. Now, calculate the partial trace over A and B in the $\sigma_x$ and $\sigma_y$ eigenstate bases.

Problem 7

Verify that in the $\sigma_z$ basis, the CNOT gate flips the first qubit iff the second qubit is spin down, i.e.

$$\begin{array}{c}
|\leftarrow\rangle \\
|\rightarrow\rangle
\end{array} \xrightarrow{\text{CNOT}} \begin{array}{c}
|\leftarrow\rangle \\
|\rightarrow\rangle
\end{array}$$

Problem 8

Find the output states of the following:

1. $\begin{array}{c}
|\otimes\rangle \\
|\otimes\rangle
\end{array}$

2. $\begin{array}{c}
|\ominus\rangle \\
|\ominus\rangle
\end{array}$

Problem 9

Verify that the output of the following diagram is $|\phi\rangle_{AB} = |\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B$

$$\begin{array}{c}
|\uparrow\rangle \\
|\downarrow\rangle
\end{array} \xrightarrow{\text{CNOT}} \begin{array}{c}
|\uparrow\rangle \\
|\downarrow\rangle
\end{array}$$

Then show that $|\phi\rangle_{AB} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A |\uparrow\rangle_B + |\downarrow\rangle_A |\downarrow\rangle_B) = \frac{1}{\sqrt{2}} (|\rightarrow\rangle_A |\rightarrow\rangle_B + |\leftarrow\rangle_A |\leftarrow\rangle_B)$.

Problem 10

Verify that the eigenvalues and corresponding eigenvectors of $\sigma_\theta = \cos \theta \sigma_z + \sin \theta \sigma_x$ are +1, -1 and $|\uparrow\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}$ respectively.
Problem 11

Verify that $\frac{1}{\sqrt{2}}(|↑⟩|↑⟩ + |↓⟩|↓⟩) = \frac{1}{\sqrt{2}}(-|↗⟩|↗⟩ + |↙⟩|↙⟩)$.

Problem 12

A pair of qubits was shared between Alice (A) and Bob (B). The qubit pair is in state $|φ⟩_{AB} = \frac{1}{\sqrt{2}}(|↑⟩_A |↑⟩_B + |↓⟩_A |↓⟩_B) = \frac{1}{\sqrt{2}}(|→⟩_A |→⟩_B + |←⟩_A |←⟩_B)$. Calculate the probability that Alice measures $|↑⟩$ and Bob measures $|↗⟩$, $P(|↑⟩_A |↗⟩_B) = \text{tr}(|↑⟩_A ⟨↑|⊗|↗⟩_B ⟨⟨⟩| |φ⟩_{AB} ⟨φ⟩)$. Then do the same for $P(|↑⟩_A |↙⟩_B)$, $P(|↓⟩_A |↗⟩_B)$, and $P(|↓⟩_A |↙⟩_B)$. 

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