2.111J/18.435J/ESD.79 Quantum Computation — Due: October 2, in class

Problem Set 3

Problem 1

Find the phase factor ϕ in the following diagram

$$|0\rangle \; — \boxed{e^{i\frac{\pi}{4}\sigma_y}} — \; \frac{e^{i\phi}}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Problem 2

Show how to construct CCNOT from CNOT and single-qubit rotations (up to an overall phase). Suggestion: Use google to search for a solution, but keep a list of what websites were used.

Problem 3

Assume that a singlet state is shared between Alice and Bob. Verify that if we measure $|0\rangle, |1\rangle$ on each qubit separately, we obtain the following probabilities of outcomes:

$$\begin{aligned} P(01) &= P(10) = \langle\psi| \left(|0\rangle_A \langle 0| \otimes |1\rangle_B \langle 1|\right) |\psi\rangle = \frac{1}{2} \\ P(11) &= 0 \end{aligned}$$

Problem 4

Verify this:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = +1 |{\rightarrow}\rangle \langle{\rightarrow}| - 1 |{\leftarrow}\rangle \langle{\leftarrow}|$$

Problem 5

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle_A |{\downarrow}\rangle_B - |{\downarrow}\rangle_A |{\uparrow}\rangle_B)$$

Rewrite $|\psi\rangle$ in terms of $|{\rightarrow}\rangle_A |{\rightarrow}\rangle_B$, $|{\leftarrow}\rangle_A |{\rightarrow}\rangle_B$, $|{\rightarrow}\rangle_A |{\leftarrow}\rangle_B$, and $|{\leftarrow}\rangle_A |{\leftarrow}\rangle_B$, where for e.g. $|{\uparrow}\rangle = \frac{1}{\sqrt{2}}(|{\rightarrow}\rangle - |{\leftarrow}\rangle)$ etc.

Problem 6

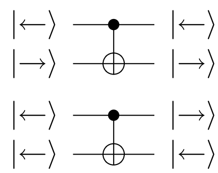
We find that

$$\mathrm{tr}_B \left|\psi\right\rangle_{AB} \left\langle\psi\right| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, where $\left|\psi\right\rangle_{AB}$ has been defined in Problem 5. This tells us that state of A looks completely random, i.e. there is a 50% chance of it being up and 50% chance of it being down. In class we have calculated this in the σ_z eigenstate basis. Now, calculate the partial trace over A and B in the σ_x and σ_y eigenstate bases.

Problem 7

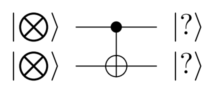
Verify that in the σ_x basis, the CNOT gate flips the first qubit iff the second qubit is spin down, i.e.



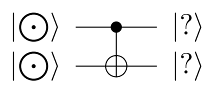
Problem 8

Find the output states of the following:

1.

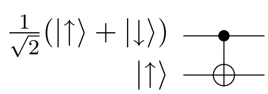


2.



Problem 9

Verify that the output of the following diagram is $\left|\phi\right\rangle_{AB} = \left|\uparrow\right\rangle_A \otimes \left|\uparrow\right\rangle_B + \left|\downarrow\right\rangle_A \otimes \left|\downarrow\right\rangle_B$



Then show that $\left|\phi\right\rangle_{AB} = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle_A \left|\uparrow\right\rangle_B + \left|\downarrow\right\rangle_A \left|\downarrow\right\rangle_B) = \frac{1}{\sqrt{2}}(\left|\rightarrow\right\rangle_A \left|\rightarrow\right\rangle_B + \left|\leftarrow\right\rangle_A \left|\leftarrow\right\rangle_B)$.

Problem 10

Verify that the eigenvalues and corresponding eigenvectors of $\sigma_\theta = \cos\theta\sigma_z + \sin\theta\sigma_x$ are +1, -1 and $\left|\nearrow\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$, $\left|\swarrow\right\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$ respectively.

Problem 11

Verify that $\frac{1}{\sqrt{2}}(|{\uparrow}\rangle\,|{\uparrow}\rangle + |{\downarrow}\rangle\,|{\downarrow}\rangle) = \frac{1}{\sqrt{2}}(-\,|{\nearrow}\rangle\,|{\nearrow}\rangle + |{\swarrow}\rangle\,|{\swarrow}\rangle)$.

Problem 12

A pair of qubits was shared between Alice (A) and Bob (B). The qubit pair is in state $|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle_A\,|{\uparrow}\rangle_B + |{\downarrow}\rangle_A\,|{\downarrow}\rangle_B) = \frac{1}{\sqrt{2}}(|{\rightarrow}\rangle_A\,|{\rightarrow}\rangle_B + |{\leftarrow}\rangle_A\,|{\leftarrow}\rangle_B)$. Calculate the probability that Alice measures $|{\uparrow}\rangle$ and Bob measures $|{\nearrow}\rangle$, $P({\uparrow}_A{\nearrow}_B) = \mathrm{tr}(|{\uparrow}\rangle_A\,\langle{\uparrow}|\otimes|{\nearrow}\rangle_B\,\langle{\nearrow}|\,|\phi\rangle_{AB}\,\langle\phi|)$. Then do the same for $P({\uparrow}_A{\swarrow}_B)$, $P({\downarrow}_A{\nearrow}_B)$, and $P({\downarrow}_A{\swarrow}_B)$.