## 2.111J/18.435J Quantum Computation Problem Set 3

(Due: Tuesday, October 4, 2005)

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*Notes for Problem Set*: As has been our convention in lecture,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote the eigenstates of $\sigma^z$ and $\sigma^x$ as follows:

$$\text{Eigenstates of } \sigma^x\text{: } |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ has eigenvalue 1. } |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ has eigenvalue -1.}$$

$$\text{Eigenstates of } \sigma^z\text{: } |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ has eigenvalue 1. } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ has eigenvalue -1.}$$

We use the following shorthand for tensor product states of two qubits named $A$ and $B$:

$$|\psi\varphi\rangle \equiv |\psi\rangle_A \otimes |\varphi\rangle_B$$

For example, the possible tensor products of two $\sigma^z$ eigenstates are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, and those of two $\sigma^x$ eigenstates are $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

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1) Show that CNOT (that is, the gate that maps $|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, and $|11\rangle \to |10\rangle$) is equivalent to

$$M_{AB} = \frac{1}{2}\left(\mathbb{I}_A + \sigma^z_A\right) \otimes \mathbb{I}_B + \frac{1}{2}\left(\mathbb{I}_A - \sigma^z_A\right) \otimes \sigma^x_B$$

by explicitly writing $M_{AB}$ in the basis that diagonalizes $\sigma^z_A \otimes \sigma^z_B$.

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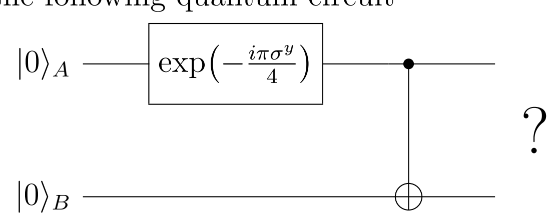
2) Evaluate the output of the following quantum circuit




How does the output compare to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$?

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3) Given two qubits in the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, what are the probabilities of measuring $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$?

4) In the basis that diagonalizes $\sigma^z_A \otimes \sigma^z_B$, write out the $4 \times 4$ unitary matrix $U$ that maps an arbitrary tensor product state $|\psi\rangle_A \otimes |\varphi\rangle_B$ to $|\varphi\rangle_A \otimes |\psi\rangle_B$. Verify that $U^2 = \mathbb{I}$.

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5) What is the output of CNOT (that is, the gate that maps $|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, and $|11\rangle \to |10\rangle$) when $|++\rangle$ is the input? When $|+-\rangle$ is the input? When $|-+\rangle$ is the input? When $|--\rangle$ is the input?

*Hint*: One way to solve this problem is simply to write out $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

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6) For a system of two qubits in the singlet state

$$\rho_{AB} = \frac{1}{2}(|01\rangle + 10\rangle)(\langle 01| + \langle 10|)$$

verify that the reduced density matrices for each qubit separately are

$$\rho_A = \rho_B = \frac{1}{2}\mathbb{I}.$$

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7) Let $\sigma^\theta \equiv \cos\theta\sigma^z + \sin\theta\sigma^x$. Verify that

$$|+_\theta\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad |-_\theta\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

where $|+_\theta\rangle$ is the eigenstate of $\sigma^\theta$ with eigenvalue $+1$, and $|-_\theta\rangle$ is the eigenstate of $\sigma^\theta$ with eigenvalue $-1$.

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8) Using the definitions of $|+_\theta\rangle$ and $|-_\theta\rangle$ from Problem 7, verify that for any $\theta$,

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|+_\theta +_\theta\rangle + |-_\theta -_\theta\rangle\right).$$

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9) Consider a system in the state $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$. Let $\text{Prob}(|\psi\phi\rangle)$ denote the probability of measuring this system to be in the state $|\psi\phi\rangle$. Once again using the definitions of $|+_\theta\rangle$ and $|-_\theta\rangle$ from Problem 7, verify that

$$\text{Prob}(|0\ +_\theta\rangle) = \text{Prob}(|1\ -_\theta\rangle) = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$\text{Prob}(|0\ -_\theta\rangle) = \text{Prob}(|1\ +_\theta\rangle) = \frac{1}{2}\sin^2\frac{\theta}{2}.$$