Notes for Problem Set: As has been our convention in lecture,

\[ \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

We denote the eigenstates of \( \sigma^z \) and \( \sigma^x \) as follows:

Eigenstates of \( \sigma^x \):

- \( |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) has eigenvalue 1
- \( |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) has eigenvalue -1.

Eigenstates of \( \sigma^z \):

- \( |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) has eigenvalue 1
- \( |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) has eigenvalue -1.

We use the following shorthand for tensor product states of two qubits named A and B:

\[ |\psi\varphi\rangle \equiv |\psi\rangle_A \otimes |\varphi\rangle_B \]

For example, the possible tensor products of two \( \sigma^z \) eigenstates are \{\( |00\rangle, |01\rangle, |10\rangle, |11\rangle \}\}, and those of two \( \sigma^x \) eigenstates are \{\( |+\rangle, |-\rangle, |+\rangle, |+\rangle \}\}.

1) Show that CNOT (that is, the gate that maps \( |00\rangle \rightarrow |00\rangle \), \( |01\rangle \rightarrow |01\rangle \), \( |10\rangle \rightarrow |11\rangle \), and \( |11\rangle \rightarrow |10\rangle \)) is equivalent to

\[ M_{AB} = \frac{1}{2} (I_A + \sigma^z_A) \otimes I_B + \frac{1}{2} (I_A - \sigma^z_A) \otimes \sigma^x_B \]

by explicitly writing \( M_{AB} \) in the basis that diagonalizes \( \sigma^z_A \otimes \sigma^z_B \).

2) Evaluate the output of the following quantum circuit

\[ |0\rangle_A \xrightarrow{\exp(-i\pi\sigma^y/4)} \]

\[ |0\rangle_B \xrightarrow{\text{CNOT}} \]

How does the output compare to \( \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)?

3) Given two qubits in the state \( |\psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \), what are the probabilities of measuring \( |+\rangle, |+\rangle, |-\rangle, |+\rangle \), and \( |-\rangle \)?
4) In the basis that diagonalizes $\sigma^z_A \otimes \sigma^z_B$, write out the $4 \times 4$ unitary matrix $U$ that maps an arbitrary tensor product state $|\psi\rangle_A \otimes |\varphi\rangle_B$ to $|\varphi\rangle_A \otimes |\psi\rangle_B$. Verify that $U^2 = I$.

5) What is the output of CNOT (that is, the gate that maps $|00\rangle \rightarrow |00\rangle$, $|01\rangle \rightarrow |01\rangle$, $|10\rangle \rightarrow |11\rangle$, and $|11\rangle \rightarrow |10\rangle$) when $|+\rangle$ is the input? When $|+\rangle$ is the input? When $|\pm\rangle$ is the input? When $|\mp\rangle$ is the input?

Hint: One way to solve this problem is simply to write out $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

6) For a system of two qubits in the singlet state

$$\rho_{AB} = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$$

verify that the reduced density matrices for each qubit separately are

$$\rho_A = \rho_B = \frac{1}{2}I.$$ 

7) Let $\sigma^\theta \equiv \cos \theta \sigma^z + \sin \theta \sigma^x$. Verify that

$$|+\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

where $|+\rangle$ is the eigenstate of $\sigma^\theta$ with eigenvalue $+1$, and $|\rangle$ is the eigenstate of $\sigma^\theta$ with eigenvalue $-1$.

8) Using the definitions of $|+\rangle$ and $|\rangle$ from Problem 7, verify that for any $\theta$,

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|+\rangle + |\rangle).$$

9) Consider a system in the state $\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$. Let Prob$(|\psi\rangle)$ denote the probability of measuring this system to be in the state $|\psi\rangle$. Once again using the definitions of $|+\rangle$ and $|\rangle$ from Problem 7, verify that

$$\text{Prob}(|0+\rangle) = \text{Prob}(|1-\rangle) = \frac{1}{2} \cos^2 \frac{\theta}{2}$$

$$\text{Prob}(|0-\rangle) = \text{Prob}(|1+\rangle) = \frac{1}{2} \sin^2 \frac{\theta}{2}.$$