2.111J/18.435J Quantum Computation Problem Set 5
(Due: Tuesday, October 25, 2005)

1) Consider the quantum phase estimation algorithm for an operator U when the algorithm's input is a particular eigenvector of U. Let $|\psi_\theta\rangle$ denote the eigenvector and let $e^{i\theta}$ denote its eigenvalue. At the end of the algorithm, right before measurement, the computer's state will be:

$$|\psi\rangle_{output} = \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} \left(\sum_{j=0}^{2^n-1} e^{-2\pi i \left(\frac{k}{2^n} - \frac{\theta}{2\pi}\right)} \right) |k\rangle \otimes |\psi_\theta\rangle$$

where $|k\rangle$ denotes the state of the n-qubit register that contains the eigenvalue estimate. Explicitly evaluate the geometric sum over j appearing in parentheses to make quantitative the following statement:

With very high probability, measurement of the n-qubit register will yield only those integers k such that $2\pi k/2^n \approx \theta$.

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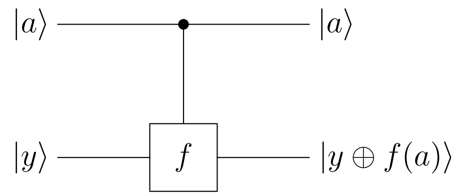
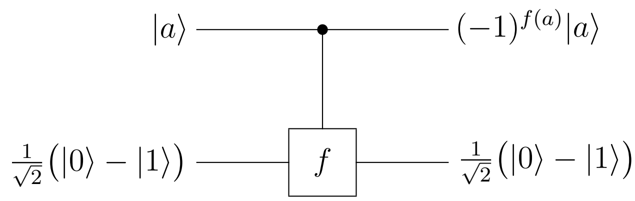
2) Verify that if one has a controlled-f gate,



then the following trick is valid:



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3) Let $H^{\otimes n}$ denote Hadamard gates applied individually to n qubits. Let $P = 2|0\rangle\langle 0| - I$ where I denotes the identity on n qubits and $|0\rangle\langle 0|$ denotes the projector onto the n-qubit state $\otimes_{i=1}^{n} |0\rangle_i$. Prove that

$$H^{\otimes n} P H^{\otimes n} = 2|\psi_u\rangle\langle\psi_u| - I$$

where $|\psi_u\rangle = \frac{1}{2^{n/2}} \sum_{j=0}^{2^n-1} |j\rangle$, the uniform superposition over the computational basis states.

4) Let O be the oracle operator for the unsorted search problem with a single search target $|w\rangle$ among N possibilities, That is, let $O = I - 2|w\rangle\langle w|$. Additionally, let $|r\rangle = \frac{1}{\sqrt{N-1}}\sum_{a\neq w}|a\rangle$ denote the uniform superposition over all the possibilities other than the search target. Define $H^{\otimes n}$ and P as in Problem 3. Finally, let $G = OH^{\otimes n}PH^{\otimes n}$ be the Grover operator. Prove that

$$G = \cos\theta\big(|w\rangle\langle w| + |r\rangle\langle r|\big) - \sin\theta\big(|w\rangle\langle r| - |r\rangle\langle w|\big) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\cos\theta = 1 - \dfrac{2}{N}$.

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5) Verify that

$$GG\ldots G = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

where $GG\ldots G$ denotes the product of k Grover operators G, defined as in Problem 4.

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6) Referring to the definitions of Problems 3 and 4, verify that

$$|\psi_u\rangle = \sqrt{\frac{1}{N}}\,|w\rangle + \sqrt{1 - \frac{1}{N}}\,|r\rangle = \begin{pmatrix} \sqrt{\frac{1}{N}} \\ \sqrt{1 - \frac{1}{N}} \end{pmatrix}.$$