# 2.111J/18.435J Quantum Computation Problem Set 7

(Due: Tuesday, November 15, 2005)

1) Consider a 2 spin system possessing an Ising interaction $\hbar\Gamma(\sigma_z^A \otimes \sigma_z^B)/2$ that is placed in a magnetic field $B(t) = B\cos(\omega t)\hat{x} + B\sin(\omega t)\hat{y} + B_0\hat{z}$. Let $\omega_A = \gamma_A B_0$ and $\omega_B = \gamma_B B_0$ denote the respective Larmor frequencies of spins $A$ and $B$ around the $z$-axis. Viewed in a frame co-rotating with the circularly polarized field, the system's equation of motion is

$$\begin{aligned}\frac{d|\chi(t)\rangle}{dt} &= -\frac{i}{\hbar}H_{rot}|\chi(t)\rangle \\ &= -\frac{i}{2}\left[(\omega_A-\omega)\sigma_z^A + (\omega_B-\omega)\sigma_z^B + \Gamma(\sigma_z^A\otimes\sigma_z^B) + \gamma_A B\sigma_x^A + \gamma_B B\sigma_x^B\right]|\chi(t)\rangle.\end{aligned}$$

Evaluate $e^{-itH_{rot}/\hbar}$ for the case where the time-dependent part of the $B$ field is applied at frequency $\omega = \omega_A - \Gamma$ for a time $t = \pi/(\gamma_A B)$. To what type of quantum logic operation does such time evolution correspond?

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2) Compare the effect of the sequence of rotations

$$e^{i\pi\sigma_y^A/4}\, e^{-i\pi(\sigma_z^A\otimes\sigma_z^B)/4}\, e^{-i\pi\sigma_x^A/4}$$

to that of a standard CNOT gate.

[*Hint*: The formula $e^{ikM} = (\cos k)\mathbb{I} + i(\sin k)M$ holds for any matrix $M$ such that $M^2 = \mathbb{I}$, not just $2\times 2$ matrices.]

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3) Consider the 3 spin Hamiltonian

$$H = -\frac{\hbar}{2}\Big[\omega_A\sigma_z^A + \omega_B\sigma_z^B + \omega_C\sigma_z^C + \Gamma_{AB}(\sigma_z^A\otimes\sigma_z^B) + \Gamma_{AC}(\sigma_z^A\otimes\sigma_z^C) + \Gamma_{BC}(\sigma_z^B\otimes\sigma_z^C)\Big].$$

The presence of the interactions splits each of the individual NMR peaks ("singlets") at $\omega_A$, $\omega_B$, and $\omega_C$ that would occur in the noninteracting case into 4 peaks (a "doublet of doublets"). Derive formulas for the size of splittings of the $\omega_A$ peak in terms of $\Gamma_{AB}, \Gamma_{AC}$, and/or $\Gamma_{BC}$.

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4) Consider the quantized simple harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2.$$

Define the annihilation operator $a$ and creation operator $a^\dagger$,

$$a \equiv \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip); \quad a^\dagger \equiv \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip).$$

Verify the following four facts.

- $[a, a^\dagger] = \mathbb{I}$.
- $[a, a^\dagger a] = a$.
- $[a^\dagger, a^\dagger a] = -a^\dagger$.
- $H = \frac{\hbar\omega}{2}\left(aa^\dagger + a^\dagger a\right) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$.

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5) Let $|0\rangle$ denote the vacuum state for the quantized simple harmonic oscillator.

$$|0\rangle \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \int dx\; e^{-m\omega x^2/(2\hbar)}\,|x\rangle = \left(\frac{1}{\pi\hbar m\omega}\right)^{1/4} \int dp\; e^{-p^2/(2m\hbar\omega)}\,|p\rangle.$$

Verify that $a|0\rangle = 0$.

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6) Define the non-normalized $n$th excited state of the quantized simple harmonic oscillator to be $|n\rangle \equiv (a^\dagger)^n|0\rangle$. Verify the following four facts.

- $a|n\rangle = \sqrt{n}|n-1\rangle$.
- $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.
- $H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$.
- $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ is the proper normalization.

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7) Consider the following form of the Jaynes-Cummings Hamiltonian,

$$H = \hbar(C + D) + \frac{\hbar\omega}{2},$$

where $C \equiv \omega\left(a^\dagger a + \frac{1}{2}\sigma_z\right)$ and $D \equiv \kappa(a^\dagger\sigma_- + a\sigma_+) - \frac{\omega - \omega_0}{2}\sigma_z$. Verify that $[C, D] = 0$.