1) Consider a 2 spin system possessing an Ising interaction $\hbar \Gamma (\sigma_z^A \otimes \sigma_z^B) / 2$ that is placed in a magnetic field $B(t) = B \cos(\omega t) \hat{x} + B \sin(\omega t) \hat{y} + B_0 \hat{z}$. Let $\omega_A = \gamma_A B_0$ and $\omega_B = \gamma_B B_0$ denote the respective Larmor frequencies of spins $A$ and $B$ around the $z$-axis. Viewed in a frame co-rotating with the circularly polarized field, the system’s equation of motion is

$$\frac{d}{dt} |\chi(t)\rangle = -\frac{i}{\hbar} H_{\text{rot}} |\chi(t)\rangle = -\frac{i}{2} \left[ (\omega_A - \omega) \sigma_z^A + (\omega_B - \omega) \sigma_z^B + \Gamma (\sigma_z^A \otimes \sigma_z^B) + \gamma_A B \sigma_x^A + \gamma_B B \sigma_x^B \right] |\chi(t)\rangle.$$ 

Evaluate $e^{-i t H_{\text{rot}} / \hbar}$ for the case where the time-dependent part of the $B$ field is applied at frequency $\omega = \omega_A - \Gamma$ for a time $t = \pi / (\gamma_A B)$. To what type of quantum logic operation does such time evolution correspond?

2) Compare the effect of the sequence of rotations $e^{i \pi \sigma_z^A / 4} e^{-i \pi (\sigma_z^A \otimes \sigma_z^B) / 4} e^{-i \pi \sigma_x^A / 4}$ to that of a standard CNOT gate.

[Hint: The formula $e^{i k M} = (\cos k) \mathbb{I} + i (\sin k) M$ holds for any matrix $M$ such that $M^2 = \mathbb{I}$, not just $2 \times 2$ matrices.]

3) Consider the 3 spin Hamiltonian

$$H = -\frac{\hbar}{2} \left[ \omega_A \sigma_z^A + \omega_B \sigma_z^B + \omega_C \sigma_z^C + \Gamma_{AB} (\sigma_z^A \otimes \sigma_z^B) + \Gamma_{AC} (\sigma_z^A \otimes \sigma_z^C) + \Gamma_{BC} (\sigma_z^B \otimes \sigma_z^C) \right].$$

The presence of the interactions splits each of the individual NMR peaks (“singlets”) at $\omega_A$, $\omega_B$, and $\omega_C$ that would occur in the noninteracting case into 4 peaks (a “doublet of doublets”). Derive formulas for the size of splittings of the $\omega_A$ peak in terms of $\Gamma_{AB}$, $\Gamma_{AC}$, and/or $\Gamma_{BC}$.

4) Consider the quantized simple harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2.$$ 

Define the annihilation operator $a$ and creation operator $a^\dagger$,

$$a \equiv \frac{1}{\sqrt{2 m \hbar \omega}} (m \omega x + i p); \quad a^\dagger \equiv \frac{1}{\sqrt{2 m \hbar \omega}} (m \omega x - i p).$$
Verify the following four facts.

- \([a, a^\dagger] = I\).
- \([a, a^\dagger a] = a\).
- \([a^\dagger, a^\dagger a] = -a^\dagger\).
- \(H = \frac{\hbar \omega}{2} (aa^\dagger + a^\dagger a) = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right)\).

5) Let \(|0\rangle\) denote the vacuum state for the quantized simple harmonic oscillator.

\[
|0\rangle \equiv \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \int dx \ e^{-m \omega x^2 / (2 \hbar)} |x\rangle = \left( \frac{1}{\pi \hbar \omega} \right)^{1/4} \int dp \ e^{-p^2 / (2 m \omega)} |p\rangle.
\]

Verify that \(a|0\rangle = 0\).

6) Define the non-normalized \(n\)th excited state of the quantized simple harmonic oscillator to be \(|n\rangle \equiv (a^\dagger)^n|0\rangle\). Verify the following four facts.

- \(a|n\rangle = \sqrt{n}|n-1\rangle\).
- \(a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\).
- \(H|n\rangle = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle\).
- \(|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n|0\rangle\) is the proper normalization.

7) Consider the following form of the Jaynes-Cummings Hamiltonian,

\[
H = \hbar(C + D) + \frac{\hbar \omega}{2},
\]

where \(C \equiv \omega \left( a^\dagger a + \frac{1}{2} \sigma_z \right)\) and \(D \equiv \kappa(a^\dagger \sigma_- + a \sigma_+) - \frac{\omega - \omega_0}{2} \sigma_z\). Verify that \([C, D] = 0\).