1) Verify that the following circuit is the appropriate encoder/decoder circuit for the 3 qubit phase flip code.

\[
\begin{array}{c}
|x\rangle \\
|0\rangle \\
|0\rangle
\end{array}
\begin{array}{c}
H \\
H \\
H
\end{array}
\quad\quad\quad\quad\quad\quad
\begin{array}{c}
\text{Phase Flip on 1 Qubit} \\
H \\
H
\end{array}
\quad\quad\quad\quad\quad\quad
\begin{array}{c}
|\psi\rangle
\end{array}
\]

In other words, exhibit a measurement on the two ancillae in the circuit’s output \(|\psi\rangle\) that will detect whether a phase flip error occurred and locate the phase flip if it did occur. Then, exhibit an operation based on the above measurement that will correct the phase flip.

2) Construct an encoding circuit and a decoding circuit for the 9 qubit Shor code

\[
|0\rangle \rightarrow \frac{1}{2\sqrt{2}} (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)
\]

\[
|1\rangle \rightarrow \frac{1}{2\sqrt{2}} (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle).
\]

3) Verify that the 9 qubit Shor code corrects a bit and/or phase flip on any 1 of the 9 qubits.

4) What is the output \(|\psi\rangle\) of the following circuit, which is the encoder for the Laflamme-Miquel-Paz-Zurek perfect 5 qubit code?

\[
\begin{array}{c}
|0\rangle \\
|0\rangle \\
|q\rangle \\
|0\rangle \\
|0\rangle
\end{array}
\begin{array}{c}
H \\
H \\
H \\
\pi \\
\pi
\end{array}
\quad\quad\quad\quad\quad\quad
\begin{array}{c}
\text{Phase Flip on 1 Qubit} \\
H \\
H
\end{array}
\quad\quad\quad\quad\quad\quad
\begin{array}{c}
|\psi\rangle
\end{array}
\]

Note the following definitions. The single qubit gates are (in the \(\sigma_z\) eigenbasis):

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad \pi = e^{i\pi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Our more complicated controlled gates, for example

1) \[
\begin{array}{c}
\frac{\pi}{2} \\
\end{array}
\]

2) \[
\begin{array}{c}
\frac{\pi}{2} \\
\end{array}
\]

3) \[
\begin{array}{c}
\frac{\pi}{2} \\
\end{array}
\]

work as follows (in the tensor product $\sigma_z$ eigenbasis):

1) does nothing to the first 3 qubits and imparts a $-1$ phase factor to the 4th qubit if and only if the first 3 qubits are in the state $|111\rangle$.

2) does nothing to the first 3 qubits and imparts a $-1$ phase factor to the 4th qubit if and only if the first 3 qubits are in the state $|010\rangle$.

3) does nothing to the 1st qubit and applies $\oplus$-gates to both the 2nd and 3rd qubits only if the 1st qubit is in the state $|1\rangle$.

5) Verify that the decoder of the above circuit allows one to correct a bit and/or phase flip on any 1 of the 5 qubits.

6) Consider the code $|0\rangle_{en} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$; $|1\rangle_{en} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Show that an arbitrary superposition of encoded states $\alpha|0\rangle_{en} + \beta|1\rangle_{en}$ is robust against errors of the form $e^{-i\theta \sigma_z/2} \otimes e^{-i\theta \sigma_z/2}$. 