2.111J/18.435J Quantum Computation Problem Set 2 Solutions
(Due: Tuesday, September 27, 2005)

Notes for Problem Set: Our convention is $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

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1) Prove that if a matrix A is Hermitian (*i.e.,* $A = A^\dagger$ where $A^\dagger$ is the matrix formed by taking the complex conjugate of each element of the transpose of A), then:

(i) All eigenvalues of A are real.

(ii) Eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Solution:

Proof of (i): Let $|a\rangle$ denote an arbitrary (and not necessarily normalized) eigenvector of A. Let a denote its eigenvalue. That is,

$$A|a\rangle = a|a\rangle$$

Left multiplying both sides of the first equation by $\langle a|$, it immediately follows that

$$\langle a|A|a\rangle = a\langle a|a\rangle.$$

However, we can also show that

$$\langle a|A|a\rangle = a^*\langle a|a\rangle$$

by the following argument:

$$\begin{aligned} A|a\rangle &= a|a\rangle && \text{(by definition)} \\ \langle a|A^\dagger &= \langle a|a^* && \text{(taking the adjoint of both sides)} \\ \langle a|A &= \langle a|a^* && \text{(since } A = A^\dagger\text{)} \\ \langle a|A|a\rangle &= a^*\langle a|a\rangle && \text{(right multiplying by } |a\rangle\text{).} \end{aligned}$$

Therefore, the fact A is Hermitian requires the eigenvalue a corresponding to an arbitrary eigenvector $|a\rangle$ must equal its complex conjugate a^* and thus be real.

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Proof of (ii): Let $|a\rangle$ and $|b\rangle$ denote an arbitrary eigenvectors of A. Let a and b denote their respective eigenvalues. That is,

$$A|a\rangle = a|a\rangle; \quad A|b\rangle = b|b\rangle;$$

Now consider the quantity $\langle a|A|b\rangle$. On the one hand, we can write

$$\langle a|A|b\rangle = \langle a|\big(A|b\rangle\big) = b\langle a|b\rangle.$$

But on the other hand, we can also write

$$\langle a|A|b\rangle = \big(\langle a|A\big)|b\rangle = a\langle a|b\rangle$$

by arguing, as we did in the proof of (i), that if A is Hermitian and $A|a\rangle = a|a\rangle$, then $\langle a|A = \langle a|a$.

Therefore, the fact A is Hermitian requires that $a\langle a|b\rangle = b\langle a|b\rangle$, and thus if the eigenvalues are not equal, $a \neq b$, then the eigenvectors must be orthogonal, $\langle a|b\rangle = 0$.

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2) Calculate the eigenvalues and normalized eigenvectors of σ_y.

3) Calculate the eigenvalues and normalized eigenvectors of $\sigma_j = \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{\sqrt{2}}\sigma_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

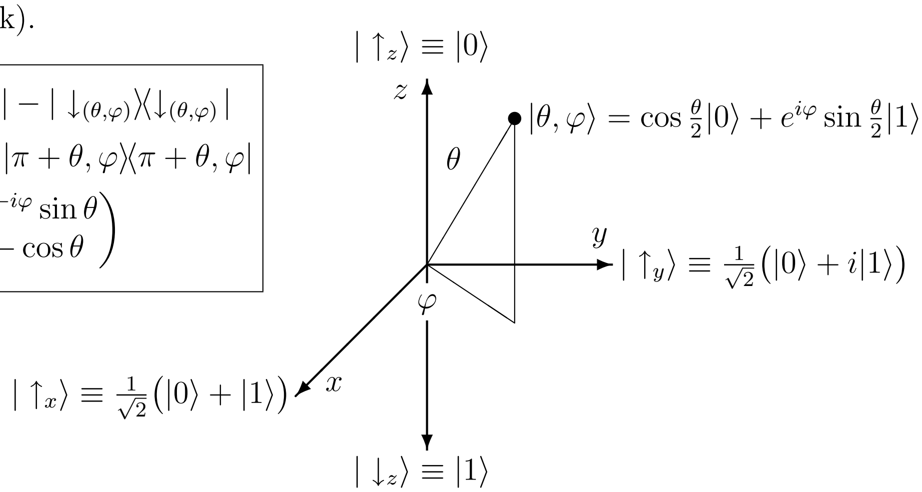
Solutions: Of course, you can solve these two problems with whatever approach you prefer from your introductory linear algebra course. However, in the quantum computation community, the Bloch sphere is a tool ubiquitously used to describe the 2×2 eigenvalue/vector problem since it nicely encapsulates how matrices of trace 0 and determinant -1 of the form

$$\sigma_{(\theta,\varphi)} \equiv \sin\theta\cos\varphi\sigma_x + \sin\theta\sin\varphi\sigma_y + \cos\theta\sigma_z$$

correspond to fundamental experimental tools such as magnets / polarizers / etc. aligned on the axis of azimuthal angle θ and polar angle φ.

Here is the Bloch sphere according to the convention used in our class (as well as Nielsen and Chuang's textbook).

$$\begin{aligned} \sigma_{(\theta,\varphi)} &\equiv |\uparrow_{(\theta,\varphi)}\rangle\langle\uparrow_{(\theta,\varphi)}| - |\downarrow_{(\theta,\varphi)}\rangle\langle\downarrow_{(\theta,\varphi)}| \\ &= |\theta,\varphi\rangle\langle\theta,\varphi| - |\pi+\theta,\varphi\rangle\langle\pi+\theta,\varphi| \\ &= \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \end{aligned}$$



As in both problem 2 and problem 3 the matrices have trace 0 and determinant -1, we know both must have eigenvalues ± 1 (as the eigenvalues summmed together must equal 0 and multiplied together must equal -1), and we know we can write them in the form $\sigma_{(\theta,\varphi)}$.

$$\sigma_y = \sigma_{(\frac{\pi}{2},\frac{\pi}{2})} \Longrightarrow \begin{cases} +1 \text{ eigenvector is } |\frac{\pi}{2},\frac{\pi}{2}\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i\pi/2}\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + i|1\rangle\big) \\ -1 \text{ eigenvector is } |\frac{3\pi}{2},\frac{\pi}{2}\rangle = \cos\frac{3\pi}{4}|0\rangle + e^{i\pi/2}\sin\frac{3\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}\big(-|0\rangle + i|1\rangle\big) \end{cases}$$

$$\sigma_j = \sigma_{(\frac{\pi}{4},0)} \Longrightarrow \begin{cases} +1 \text{ eigenvector is } |\frac{\pi}{2},0\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle = 0.9239|0\rangle + 0.3827|1\rangle \\ -1 \text{ eigenvector is } |\frac{5\pi}{8},0\rangle = \cos\frac{5\pi}{8}|0\rangle + \sin\frac{5\pi}{8}|1\rangle = -0.3827|0\rangle + 0.9239|1\rangle \end{cases}$$

With the above, problems 2 and 3 are answered. But for completeness, let's now derive these handy-dandy formulas of:

$$|\uparrow_{(\theta,\varphi)}\rangle \;=\; \cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle$$

$$\begin{aligned} \sigma_{(\theta,\varphi)} &\equiv \sin\theta\cos\varphi\sigma_x + \sin\theta\sin\varphi\sigma_y + \cos\theta\sigma_z \\ &= |\theta,\varphi\rangle\langle\theta,\varphi| - |\theta+\pi,\varphi\rangle\langle\theta+\pi,\varphi| \\ &= \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}. \end{aligned}$$

The key idea is that if the state $|\uparrow_z\rangle \equiv |0\rangle$ denotes "spin up" along the z-axis and if the operator $R_j(\alpha)$ enacts a counterclockwise rotation of angle α around axis j, then the state denoting "spin up" along the axis with azimuthal angle θ and polar angle φ will be, up to global phase factor,

$$|\theta,\varphi\rangle \equiv R_z(\varphi)R_y(\theta)|0\rangle$$

Our convention is to chose the global phase such that the scalar in front of $|0\rangle$ is always real and only the scalar in front of $|1\rangle$ can be complex. As we will see momentarily, this is achieved by writing

$$|\theta,\varphi\rangle \equiv \widetilde{R}_z(\varphi)R_y(\theta)|0\rangle \equiv e^{i\varphi/2}R_z(\varphi)R_y(\theta)|0\rangle$$

Now, recall that rotations are related to the Pauli matrices as follows

$$R_j(\alpha) \equiv \exp\left(\frac{-i\alpha\sigma_j}{2}\right) = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\sigma_j.$$

Please note our convention is that counterclockwise rotations entail negative signs in the exponent and thus negative signs in front of the sine term.

For example,

$$\begin{aligned} R_x(\alpha) &\equiv \exp\left(\frac{-i\alpha\sigma_x}{2}\right) = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\sigma_x = \begin{pmatrix} \cos\frac{\alpha}{2} & -i\sin\frac{\alpha}{2} \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \\ R_y(\alpha) &\equiv \exp\left(\frac{-i\alpha\sigma_y}{2}\right) = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\sigma_y = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}, \\ R_z(\alpha) &\equiv \exp\left(\frac{-i\alpha\sigma_z}{2}\right) = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\sigma_z = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}, \end{aligned}$$

and thus

$$\widetilde{R}_z(\alpha) \equiv e^{i\alpha/2} R_z(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}.$$

Therefore, as promised, we conclude

$$\begin{aligned} |\theta, \varphi\rangle &\equiv \widetilde{R}_z(\varphi) R_y(\theta)|0\rangle \\ &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \\ &= \cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle. \end{aligned}$$

With this formula established, one can establish the formula for $\sigma_{(\theta,\varphi)}$ either by explicitly writing out the projectors $|\theta, \varphi\rangle\langle\theta, \varphi|$ and $|\theta + \pi, \varphi\rangle\langle\theta + \pi, \varphi|$ or instead by realizing that

$$\sigma_{(\theta,\varphi)} = R_y(-\theta)\widetilde{R}_z(-\varphi)\sigma_z\widetilde{R}_z(\varphi)R_y(\theta)$$

and explicitly writing out the rotation matrices.

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4) Verify that (i) $\sigma_z\sigma_x = i\sigma_y$, (ii) $\sigma_x\sigma_z = -i\sigma_y$, (iii) $\sigma_y\sigma_z = i\sigma_x$, (iv) $\sigma_z\sigma_y = -i\sigma_x$.

Solution: First, Pauli matrices anticommute. Therefore, we have the implications (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). Then, to finish the problem by establishing, say, statements (i) and (iii), simply realize that in light of the fact Pauli matrices anticommute, the Pauli commutation relations

$$[\sigma_x, \sigma_y] = 2i\sigma_z \text{ and cyclic permutations}$$

can be rewritten as

$$2\sigma_x\sigma_y = 2i\sigma_z \text{ and cyclic permutations.}$$

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5) Prove that if j_x, j_y, and j_z are real numbers such that $j_x^2 + j_y^2 + j_z^2 = 1$, then $\sigma_j^2 = (j_x\sigma_x + j_y\sigma_y + j_z\sigma_z)^2 = \mathbb{I}$, the identity.

Solution: The key is to realize that

$$(j_x\sigma_x+j_y\sigma_y+j_z\sigma_z)^2 = j_x^2\sigma_x^2 + j_y^2\sigma_y^2 + j_z^2\sigma_z^2 + (\text{cross terms that vanish since Paulis anticommute}).$$

Then, since Paulis square to the identity and since we're given $j_x^2 + j_y^2 + j_z^2 = 1$, the desired statement $\sigma_j^2 = \mathbb{I}$ immediately follows.

6) Calculate the 2×2 unitary matrix corresponding to a rotation of π radians counterclockwise around $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. What does this rotation do to $|\uparrow\rangle, |\downarrow\rangle, |\rightarrow\rangle$, and $|\leftarrow\rangle$ (*i.e.*, the eigenvectors of σ_z and σ_x, respectively)?

Solution: By definition,

$$R_{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})}(\pi) \equiv \exp\left(\frac{-i\pi(\sigma_x+\sigma_z)}{2\sqrt{2}}\right)$$

Again, please remember our convention is counterclockwise rotations entail minus signs in the exponent.

In light of Problem 5, $[\frac{1}{\sqrt{2}}(\sigma_x+\sigma_z)]^2 = \mathbb{I}$, and thus we may use the handy-dandy formula

$$\exp(-i\theta M) = \cos\theta\mathbb{I} - i\sin\theta M$$

which, please remember, only holds for matrices M for which $M^2 = \mathbb{I}$.

Therefore,

$$R_{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})}(\pi) = \cos\left(\frac{\pi}{2}\right)\mathbb{I} - \frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2}\right)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

and

$$\begin{aligned}
R_{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})}(\pi)|\uparrow\rangle &= -\frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = -i|\rightarrow\rangle\\
R_{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})}(\pi)|\downarrow\rangle &= -\frac{i}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = -\frac{i}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = -i|\leftarrow\rangle\\
R_{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})}(\pi)|\rightarrow\rangle &= -\frac{i}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = -i\begin{pmatrix}1\\ 0\end{pmatrix} = -i|\uparrow\rangle\\
R_{(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})}(\pi)|\leftarrow\rangle &= -\frac{i}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = -i\begin{pmatrix}0\\ 1\end{pmatrix} = -i|\downarrow\rangle.
\end{aligned}$$