**2.111J/18.435J Quantum Computation Problem Set 3 Solutions**
(Due: Tuesday, October 4, 2005)

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*Notes for Problem Set*: As has been our convention in lecture,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote the eigenstates of $\sigma^z$ and $\sigma^x$ as follows:

$$\text{Eigenstates of } \sigma^x\text{: } |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ has eigenvalue 1. } |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ has eigenvalue -1.}$$

$$\text{Eigenstates of } \sigma^z\text{: } |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ has eigenvalue 1. } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ has eigenvalue -1.}$$

We use the following shorthand for tensor product states of two qubits named $A$ and $B$:

$$|\psi\varphi\rangle \equiv |\psi\rangle_A \otimes |\varphi\rangle_B$$

For example, the possible tensor products of two $\sigma^z$ eigenstates are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, and those of two $\sigma^x$ eigenstates are $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

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1) Show that CNOT (that is, the gate that maps $|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, and $|11\rangle \to |10\rangle$) is equivalent to

$$M_{AB} = \frac{1}{2}(\mathbb{I}_A + \sigma^z_A) \otimes \mathbb{I}_B + \frac{1}{2}(\mathbb{I}_A - \sigma^z_A) \otimes \sigma^x_B$$

by explicitly writing $M_{AB}$ in the basis that diagonalizes $\sigma^z_A \otimes \sigma^z_B$.

**Solution:** Given the input-output relations of a reversible gate, one can immediately write the unitary matrix $U$ representing that gate as a sum of projectors of the form

$$U = \sum_i |Output_i\rangle\langle Input_i|$$

Thus,

$$U_{\text{CNOT}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

[NB: Our matrix representation uses our usual convention for the basis that diagonalizes $\sigma^z_A \otimes \sigma^z_B$. Namely, the rows are arranged from top to bottom by counting in binary, *i.e.*, topmost row = $|00\rangle$, next row $|01\rangle$, row below that = $|10\rangle$, bottommost row = $|11\rangle$]

So the question now is:

$$M_{AB} = \frac{1}{2}\left(\mathbb{I}_A + \sigma_A^z\right)\otimes \mathbb{I}_B + \frac{1}{2}\left(\mathbb{I}_A - \sigma_A^z\right)\otimes \sigma_B^x \stackrel{?}{=} \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

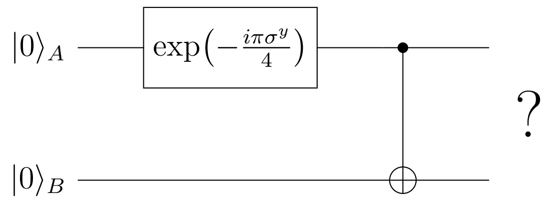
which is simply a matter of tensor product arithmetic. Given that this problem is our class's first problem centering on tensor product arithmetic, I shall present the arithmetic in full (indeed, perhaps excessive) step-by-step detail.

$$\begin{aligned}
M_{AB} &= \frac{1}{2}\left(\mathbb{I}_A + \sigma_A^z\right)\otimes \mathbb{I}_B + \frac{1}{2}\left(\mathbb{I}_A - \sigma_A^z\right)\otimes \sigma_B^x \\
&= \frac{1}{2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]_A \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}_B + \frac{1}{2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]_A \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix}_B \\
&= \begin{pmatrix}1&0\\0&0\end{pmatrix}_A \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}_B + \begin{pmatrix}0&0\\0&1\end{pmatrix}_A \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix}_B \\
&= \begin{pmatrix} 1\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{pmatrix} + \begin{pmatrix} 0\begin{pmatrix}0&1\\1&0\end{pmatrix} & 0\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ 0\begin{pmatrix}0&1\\1&0\end{pmatrix} & 1\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{pmatrix} \\
&= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \\
&= U_{\text{CNOT}}
\end{aligned}$$

Or, if you prefer to work in terms of projectors,

$$\begin{aligned}
M_{AB} &= \frac{1}{2}\left(\mathbb{I}_A + \sigma_A^z\right)\otimes \mathbb{I}_B + \frac{1}{2}\left(\mathbb{I}_A - \sigma_A^z\right)\otimes \sigma_B^x \\
&= \frac{1}{2}\Big[\big(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\big) + \big(|0\rangle\langle 0|_A - |1\rangle\langle 1|_A\big)\Big] \otimes \big(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\big) \\
&\quad + \frac{1}{2}\Big[\big(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\big) - \big(|0\rangle\langle 0|_A - |1\rangle\langle 1|_A\big)\Big] \otimes \big(|0\rangle\langle 1|_B + |1\rangle\langle 0|_B\big) \\
&= |0\rangle\langle 0|_A \otimes \big(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\big) \; + \; |1\rangle\langle 1|_A \otimes \big(|0\rangle\langle 1|_B + |1\rangle\langle 0|_B\big) \\
&= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| \\
&= U_{\text{CNOT}}
\end{aligned}$$

2) Evaluate the output of the following quantum circuit




How does the output compare to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$?

**Solution:** The circuit's input state is

$$|\psi_{\text{in}}\rangle_{AB} = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

After the first gate, the state is

$$\begin{aligned}
|\psi_1\rangle_{AB} &= \left[\exp\left(\frac{-i\pi\sigma_A^y}{4}\right) \otimes \mathbb{I}_B\right] |\psi_{\text{in}}\rangle_{AB} \\
&= \left[\left(\cos\left(\frac{\pi}{4}\right)\mathbb{I}_A - i\sin\left(\frac{\pi}{4}\right)\sigma_A^y\right) \otimes \mathbb{I}_B\right] |00\rangle \\
&= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)
\end{aligned}$$

After the second and final gate, the state is

$$\begin{aligned}|\psi_{\text{out}}\rangle_{AB} &= U_{CNOT}|\psi_1\rangle_{AB} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\end{aligned}$$

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3) Given two qubits in the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, what are the probabilities of measuring $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$?

**Solution:** *Note the solution of this problem can be viewed as a special case of the general fact proven in Problem 8. Namely,*

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|+_{\hat{j}}+_{\hat{j}}\rangle + |-_{\hat{j}}-_{\hat{j}}\rangle)$$

*where $|+_{\hat{j}}\rangle$ denotes "spin up" relative to an arbitrarily chosen axis $\hat{j}$ of the Bloch sphere and $|+_{\hat{j}}\rangle$ denotes "spin down." (In this notation, our usual notation $|\pm\rangle$ would be $|\pm_{\hat{x}}\rangle$.) However, in the solution below, we take a straightforward, specific approach.*

Recall that

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \text{ and } |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Hence,

$$\begin{aligned}|\psi\rangle_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{2\sqrt{2}}\left[(|+\rangle + |-\rangle)_A \otimes (|+\rangle + |-\rangle)_B \;+\; (|+\rangle - |-\rangle)_A \otimes (|+\rangle - |-\rangle)_B\right] \\ &= \frac{1}{2\sqrt{2}}\left[(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle) + (|++\rangle - |+-\rangle - |-+\rangle + |--\rangle)\right] \\ &= \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)\end{aligned}$$

Thus, we conclude

$$\text{Prob}(|++\rangle) = |\langle ++|\psi\rangle_{AB}|^2 = \frac{1}{2} \qquad \text{Prob}(|+-\rangle) = |\langle +-|\psi\rangle_{AB}|^2 = 0$$

$$\text{Prob}(|-+\rangle) = |\langle -+|\psi\rangle_{AB}|^2 = 0 \qquad \text{Prob}(|--\rangle) = |\langle --|\psi\rangle_{AB}|^2 = \frac{1}{2}$$

4) In the basis that diagonalizes $\sigma_A^z \otimes \sigma_B^z$, write out the $4 \times 4$ unitary matrix $U$ that maps an arbitrary tensor product state $|\psi\rangle_A \otimes |\varphi\rangle_B$ to $|\varphi\rangle_A \otimes |\psi\rangle_B$. Verify that $U^2 = \mathbb{I}$.

**Solution:** Working in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ that diagonalizes $\sigma_A^z \otimes \sigma_B^z$, we see we desire a unitary matrix $U$ such that for any complex scalars $\alpha_0, \alpha_1, \beta_0$ and $\beta_1$,

$$U\left[(\alpha_0|0\rangle_A + \alpha_1|1\rangle_A) \otimes (\beta_0|0\rangle_B + \beta_1|1\rangle_B)\right] = (\beta_0|0\rangle_A + \beta_1|1\rangle_A) \otimes (\alpha_0|0\rangle_B + \alpha_1|1\rangle_B)$$

Expanding out the tensor products, this means we desire $U$ such that

$$U(\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle) = \alpha_0\beta_0|00\rangle + \alpha_1\beta_0|01\rangle + \alpha_0\beta_1|10\rangle + \alpha_1\beta_1|11\rangle$$

Thus, we want a unitary operator that leaves the $|00\rangle$ and $|11\rangle$ components invariant and swaps the $|01\rangle$ and $|10\rangle$ components, and this action is achieved by

$$U_{\text{SWAP}} = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can immediately see that $U_{\text{SWAP}}^2 = \mathbb{I}$ since $U_{\text{SWAP}}$ is block diagonal, the only nontrivial block is of the form $\sigma^x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $(\sigma^x)^2 = \mathbb{I}$.

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5) What is the output of CNOT (that is, the gate that maps $|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$, $|10\rangle \to |11\rangle$, and $|11\rangle \to |10\rangle$) when $|++\rangle$ is the input? When $|+-\rangle$ is the input? When $|-+\rangle$ is the input? When $|--\rangle$ is the input?

*Hint*: One way to solve this problem is simply to write out $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

**Solution:** The most straightforward way to solve this problem is to take the hint and write

$$|++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \qquad |+-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$
$$|-+\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \qquad |--\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

, and then apply

$$U_{\text{CNOT}} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, we conclude

$$\begin{aligned}
U_{\rm CNOT}|++\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) = |++\rangle \\
U_{\rm CNOT}|+-\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |11\rangle - |10\rangle) = |--\rangle \\
U_{\rm CNOT}|-+\rangle &= \frac{1}{2}(|00\rangle + |01\rangle - |11\rangle - |10\rangle) = |-+\rangle \\
U_{\rm CNOT}|--\rangle &= \frac{1}{2}(|00\rangle - |01\rangle - |11\rangle + |10\rangle) = |+-\rangle
\end{aligned}$$

So we see that while in the $\sigma_A^z \otimes \sigma_B^z$ eigenbasis qubit $A$ serves as the control bit and goes through unchanged and qubit $B$ serves as the target bit and is flipped if the control bit is $|1\rangle_A$, *in the $\sigma_A^x \otimes \sigma_B^x$ eigenbasis this arrangement is reversed.* That is, in $\sigma_A^x \otimes \sigma_B^x$ eigenbasis, qubit $B$ serves as the control bit and goes through unchanged, while qubit $A$ serves as the target bit and is flipped if the control bit is $|-\rangle_B$.

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6) For a system of two qubits in the singlet state

$$\rho_{AB} = \frac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|)$$

verify that the reduced density matrices for each qubit separately are

$$\rho_A = \rho_B = \frac{1}{2}\mathbb{I}.$$

**Solution:** *Note there is a typo in the statement of the problem. A singlet has minus signs $\rho_{singlet} = \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)$, and not the plus signs of $\rho_{AB}$ above. However, in this case, the choice of relative sign makes no difference to the reduced density matrices, and we shall explicitly derive this fact below by considering $\rho_\pm = \frac{1}{2}(|01\rangle \pm |10\rangle)(\langle 01| \pm \langle 10|)$.*

Expanding out the projector,

$$\begin{aligned}
\rho_\pm &= \frac{1}{2}(|01\rangle \pm |10\rangle)(\langle 01| \pm \langle 10| \\
&= \frac{1}{2}(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|)
\end{aligned}$$

The reduced density matrix for qubit $A$ is calculated by tracing out the states of qubit $B$.

$$\begin{aligned}
\rho_{\pm,A} &= {\rm Tr}_B(\rho_{\pm,A}) \\
&= \frac{1}{2}{\rm Tr}_B(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|) \\
&= \frac{1}{2}(|0\rangle\langle 0|_A\langle 1|1\rangle_B \pm |0\rangle\langle 1|_A\langle 1|0\rangle_B \pm |1\rangle\langle 0|_A\langle 0|1\rangle_B + |1\rangle\langle 1|_A\langle 0|0\rangle_B) \\
&= \frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A) \\
&= \frac{1}{2}\mathbb{I}_A
\end{aligned}$$

since $\{|0\rangle_B, |1\rangle_B\}$ are orthonormal, $\langle i|j\rangle_B = \delta_{ij}$.

Similarly, the reduced density matrix for qubit $B$ is calculated by tracing out the states of qubit $A$.

$$\begin{aligned}
\rho_{\pm,B} &= \text{Tr}_A(\rho_{\pm,B}) \\
&= \frac{1}{2}\text{Tr}_A(|01\rangle\langle 01| \pm |01\rangle\langle 10| \pm |10\rangle\langle 01| + |10\rangle\langle 10|) \\
&= \frac{1}{2}(\langle 0|0\rangle_A |1\rangle\langle 1|_B \pm \langle 0|1\rangle_A |1\rangle\langle 0|_B \pm \langle 1|0\rangle_A |0\rangle\langle 1|_B) + \langle 1|1\rangle_A |0\rangle\langle 0|_B) \\
&= \frac{1}{2}(|1\rangle\langle 1|_B + |0\rangle\langle 0|_B) \\
&= \frac{1}{2}\mathbb{I}_B
\end{aligned}$$

since $\{|0\rangle_A, |1\rangle_A\}$ are orthonormal, $\langle i|j\rangle_A = \delta_{ij}$.

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7) Let $\sigma^\theta \equiv \cos\theta\sigma^z + \sin\theta\sigma^x$. Verify that

$$|+_\theta\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad |-_\theta\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

where $|+_\theta\rangle$ is the eigenstate of $\sigma^\theta$ with eigenvalue $+1$, and $|-_\theta\rangle$ is the eigenstate of $\sigma^\theta$ with eigenvalue $-1$.

**Solution:** Perhaps the quickest way to verify these facts is to realize $\sigma^\theta = R_{\hat{y}}(\theta)\sigma^z R_{\hat{y}}(-\theta)$ by the definition of $R_{\hat{y}}(\theta)$ as the rotation matrix for a counterclockwise rotation of $\theta$ around the $\hat{y}$-axis. Thus,

$$\begin{aligned}
|+_\theta\rangle &= R_{\hat{y}}(\theta)|0\rangle = \left[\cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\sigma^y\right]|0\rangle = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \\
|-_\theta\rangle &= R_{\hat{y}}(\theta)|1\rangle = \left[\cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\sigma^y\right]|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.
\end{aligned}$$

*Pedagogical Sidenote*: Though $\sigma^\theta = R_{\hat{y}}(\theta)\sigma^z R_{\hat{y}}(-\theta)$ must be true due to the definition of $R_{\hat{y}}(\theta)$ as the rotation matrix for a counterclockwise rotation of $\theta$ around the $\hat{y}$-axis, let us briefly digress here for an explicit verification:

$$\begin{aligned}
R_{\hat{y}}(\theta)\sigma^z R_{\hat{y}}(-\theta) &= \exp\left(\frac{-i\theta\sigma^y}{2}\right)\sigma^z \exp\left(\frac{i\theta\sigma^y}{2}\right) \\
&= \left[\cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\sigma^y\right]\sigma^z\left[\cos\left(\frac{\theta}{2}\right)\mathbb{I} + i\sin\left(\frac{\theta}{2}\right)\sigma^y\right] \\
&= \cos^2\left(\frac{\theta}{2}\right)\sigma^z + i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)[\sigma^z,\sigma^y] + \sin^2\left(\frac{\theta}{2}\right)\sigma^y\sigma^z\sigma^y \\
&= \cos^2\left(\frac{\theta}{2}\right)\sigma^z + i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)(-2i\sigma^x) + \sin^2\left(\frac{\theta}{2}\right)([\sigma^y,\sigma^z] + \sigma^z\sigma^y)\sigma^y \\
&= \cos^2\left(\frac{\theta}{2}\right)\sigma^z + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\sigma^x + \sin^2\left(\frac{\theta}{2}\right)(2i\sigma^x\sigma^y + \sigma^z) \\
&= \cos^2\left(\frac{\theta}{2}\right)\sigma^z + \sin\theta\sigma^x + \sin^2\left(\frac{\theta}{2}\right)(-\sigma^z) \\
&= \cos\theta\sigma^z + \sin\theta\sigma^x \\
&= \sigma^\theta
\end{aligned}$$

where we have used the facts $\cos^2\varphi - \sin^2\varphi = \cos(2\varphi)$, $2\sin\varphi\cos\varphi = \sin(2\varphi)$, $[\sigma^y,\sigma^z] = 2i\sigma^x$, and $(\sigma^y)^2 = \mathbb{I}$.

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8) Using the definitions of $|+_\theta\rangle$ and $|-_\theta\rangle$ from Problem 7, verify that for any $\theta$,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|+_\theta +_\theta\rangle + |-_\theta -_\theta\rangle).$$

**Solution:** Using the definitions of $|+_\theta\rangle$ and $|-_\theta\rangle$ from Problem 7,

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|+_\theta +_\theta\rangle + |-_\theta -_\theta\rangle) &= \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle\right]\otimes\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle\right] \\
&\quad + \frac{1}{\sqrt{2}}\left[-\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle\right]\otimes\left[-\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right](|00\rangle + |11\rangle) \\
&\quad + \frac{1}{\sqrt{2}}\left[\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right](|01\rangle + |10\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\end{aligned}$$

9) Consider a system in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Let Prob($|\psi\phi\rangle$) denote the probability of measuring this system to be in the state $|\psi\phi\rangle$. Once again using the definitions of $|+_\theta\rangle$ and $|-_\theta\rangle$ from Problem 7, verify that

$$\text{Prob}(|0\,+_\theta\rangle) = \text{Prob}(|1\,-_\theta\rangle) = \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)$$

$$\text{Prob}(|0\,-_\theta\rangle) = \text{Prob}(|1\,+_\theta\rangle) = \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right).$$

**Solution:**

$$\begin{aligned}
\text{Prob}(|0\,+_\theta\rangle) &= \left|\left\langle 0\ +_\theta \,\middle|\, \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)\right.\right|^2 \\
&= \frac{1}{2}\left|\left[\cos\left(\frac{\theta}{2}\right)\left\langle 00\right| + \sin\left(\frac{\theta}{2}\right)\left\langle 01\right|\right]\left[\left|00\right\rangle + \left|11\right\rangle\right]\right|^2 \\
&= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)|\langle 00|00\rangle|^2 + \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)|\langle 01|11\rangle|^2 \\
&= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Prob}(|1\,-_\theta\rangle) &= \left|\left\langle 1\ -_\theta \,\middle|\, \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)\right.\right|^2 \\
&= \frac{1}{2}\left|\left[-\sin\left(\frac{\theta}{2}\right)\left\langle 10\right| + \cos\left(\frac{\theta}{2}\right)\left\langle 11\right|\right]\left[\left|00\right\rangle + \left|11\right\rangle\right]\right|^2 \\
&= \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)|\langle 10|00\rangle|^2 + \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)|\langle 11|11\rangle|^2 \\
&= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Prob}(|0\,-_\theta\rangle) &= \left|\left\langle 0\ -_\theta \,\middle|\, \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)\right.\right|^2 \\
&= \frac{1}{2}\left|\left[-\sin\left(\frac{\theta}{2}\right)\left\langle 00\right| + \cos\left(\frac{\theta}{2}\right)\left\langle 01\right|\right]\left[\left|00\right\rangle + \left|11\right\rangle\right]\right|^2 \\
&= \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)|\langle 00|00\rangle|^2 + \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)|\langle 01|11\rangle|^2 \\
&= \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
\text{Prob}(|1+_\theta\rangle) &= \left|\left\langle 1\ +_\theta \right| \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right)\right|^2 \\
&= \frac{1}{2}\left|\left[\cos\left(\frac{\theta}{2}\right)\left\langle 10\right| + \sin\left(\frac{\theta}{2}\right)\left\langle 11\right|\right]\left[\left|00\right\rangle + \left|11\right\rangle\right]\right|^2 \\
&= \frac{1}{2}\cos^2\left(\frac{\theta}{2}\right)|\langle 10|00\rangle|^2 + \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)|\langle 11|11\rangle|^2 \\
&= \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)
\end{aligned}
$$