MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.14 Analysis and Design of Feedback Control Sysytems

The Dirac Delta Function and Convolution

1 The Dirac Delta (Impulse) Function

The Dirac delta function is a non-physical, singularity function with the following definition

$$\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \text{undefined} & \text{at } x = 0 \end{cases} \tag{1}$$

but with the requirement that

$$\int_{-\infty}^{\infty} \delta(x) dx = 1, \tag{2}$$

that is, the function has unit area.

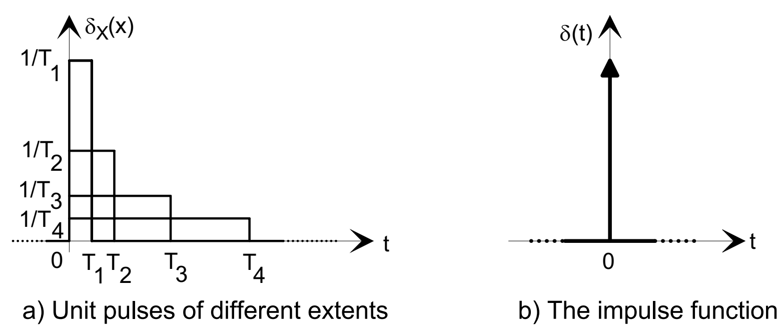

Figure 1: Unit pulses and the Dirac delta function.

Figure 1 shows a *unit pulse* function $\delta_T(t)$, that is a brief rectangular pulse function of duration T, defined to have a constant amplitude $1/T$ over its extent, so that the area $T \times 1/T$ under the pulse is unity:

$$\delta_T(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1/T & 0 < t \leq T \\ 0 & \text{for } t > 0. \end{cases} \tag{3}$$

The Dirac delta function (also known as the impulse function) can be defined as the limiting form of the unit pulse $\delta_T(t)$ as the duration T approaches zero. As the duration T of $\delta_T(t)$ decreases, the amplitude of the pulse increases to maintain the requirement of unit area under the function, and

$$\delta(t) = \lim_{T \to 0} \delta_T(t). \tag{4}$$

The impulse is therefore defined to exist only at time $t = 0$, and although its value is strictly undefined at that time, it must tend toward infinity so as to maintain the property of unit area in the limit. The *strength* of a scaled impulse $K\delta(t)$ is defined by its area K.

The limiting form of many other functions may be used to approximate the impulse. Common functions include triangular, gaussian, and sinc ($\sin(x)/x$) functions.

The impulse function is used extensively in the study of linear systems, both spatial and temporal. Although true impulse functions are not found in nature, they are approximated by short duration, high amplitude phenomena such as a hammer impact on a structure, or a lightning strike on a radio antenna. As we will see below, the response of a causal linear system to an impulse defines its response to all inputs.

An impulse occurring at $t = a$ is $\delta(t - a)$.

1.1 The "Sifting" Property of the Impulse

When an impulse appears in a product within an integrand, it has the property of "sifting" out the value of the integrand at the point of its occurrence:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{5}$$

This is easily seen by noting that $\delta(t - a)$ is zero except at $t = a$, and for its infinitesimal duration $f(t)$ may be considered a constant and taken outside the integral, so that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)\int_{-\infty}^{\infty} \delta(t-a)dt = f(a) \tag{6}$$

from the unit area property.

2 Convolution

Consider a linear continuous-time system with input $u(t)$, and response $y(t)$, as shown in Fig. 2. We assume that the system is initially at rest, that is all initial conditions are zero at time $t = 0$, and examine the time-domain forced response $y(t)$ to a continuous input waveform $u(t)$.

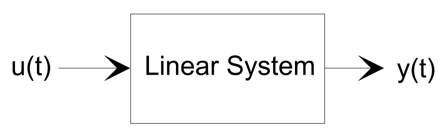

Figure 2: A linear system.

In Fig. 3 an arbitrary continuous input function $u(t)$ has been approximated by a *staircase* function $\tilde{u}_T(t) \approx u(t)$, consisting of a series of *piecewise constant* sections each of an arbitrary fixed duration, T, where

$$\tilde{u}_T(t) = u(nT) \qquad \text{for } nT \le t < (n+1)T \tag{7}$$

for all n. It can be seen from Fig. 3 that as the interval T is reduced, the approximation becomes more exact, and in the limit

$$u(t) = \lim_{T \to 0} \tilde{u}_T(t).$$

The staircase approximation $\tilde{u}_T(t)$ may be considered to be a sum of non-overlapping delayed pulses $p_n(t)$, each with duration T but with a different amplitude $u(nT)$:

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} p_n(t) \tag{8}$$

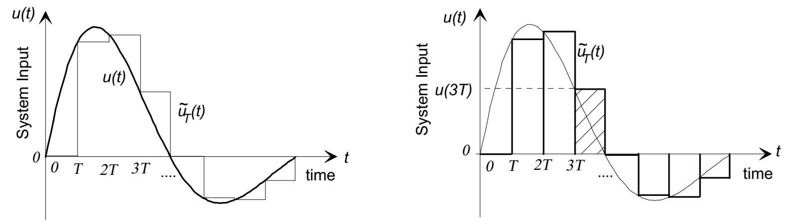

Figure 3: Staircase approximation to a continuous input function $u(t)$.

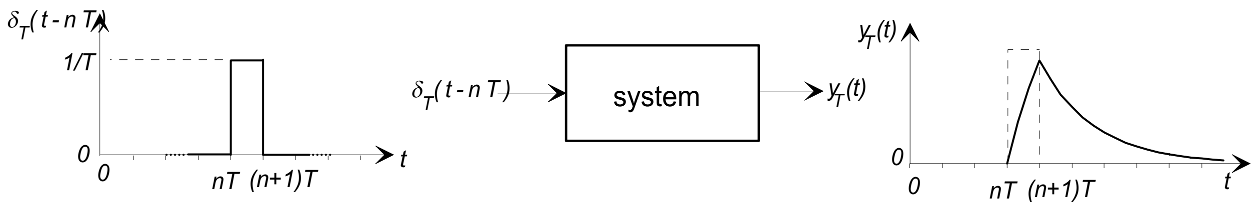

Figure 4: System response to a unit pulse of duration T.

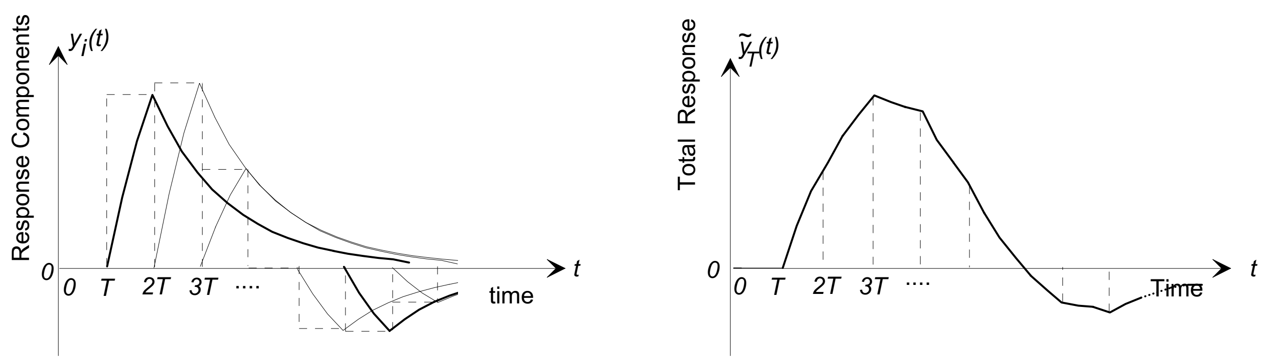

Figure 5: System response to individual pulses in the staircase approximation to $u(t)$.

where

$$p_n(t) = \begin{cases} u(nT) & nT \le t < (n+1)T \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Each component pulse $p_n(t)$ may be written in terms of a delayed unit pulse $\delta_T(t)$ defined in Sec. 1, that is:

$$p_n(t) = u(nT)\delta_T(t - nT)T \tag{10}$$

so that Eq. (8) may be written:

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)\delta_T(t - nT)T. \tag{11}$$

We now assume that the system response to $\delta_T(t)$ is a known function and is designated $h_T(t)$ as shown in Fig. 4. Then if the system is linear and time-invariant, the response to a delayed unit pulse, occurring at time nT, is simply a delayed version of the pulse response:

$$y_n(t) = h_T(t - nT). \tag{12}$$

The principle of superposition allows the total system response to $\tilde{u}_T(t)$ to be written as the sum of the responses to all of the component weighted pulses in Eq. (11):

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)h_T(t - nT)T \tag{13}$$

as shown in Fig. 5. For physical systems the pulse response $h_T(t)$ is zero for time $t < 0$, and future components of the input do not contribute to the sum, so that the upper limit of the summation may be rewritten:

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{N} u(nT)h_T(t - nT)T \qquad \text{for } NT \le t < (N+1)T. \tag{14}$$

Equation (14) expresses the system response to the staircase approximation of the input in terms of the system pulse response $h_T(t)$. If we now let the pulse width T become very small, and write $nT = \tau$, $T = d\tau$, and note that $\lim_{T \to 0} \delta_T(t) = \delta(t)$, the summation becomes an integral:

$$y(t) = \lim_{T \to 0} \sum_{n=-\infty}^{N} u(nT)h_T(t - nT)T \tag{15}$$

$$= \int_{-\infty}^{t} u(\tau)h(t - \tau)d\tau \tag{16}$$

where $h(t)$ is defined to be the system *impulse response*,

$$h(t) = \lim_{T \to 0} h_T(t). \tag{17}$$

Equation (16) is an important integral in the study of linear systems and is known as the *convolution* or *superposition* integral. It states that the system is entirely *characterized* by its response to an impulse function $\delta(t)$, in the sense that the forced response to any arbitrary input $u(t)$ may be computed from knowledge of the impulse response alone. The convolution operation is often written using the symbol $\otimes$:

$$y(t) = u(t) \otimes h(t) = \int_{-\infty}^{t} u(\tau)h(t - \tau)d\tau. \tag{18}$$

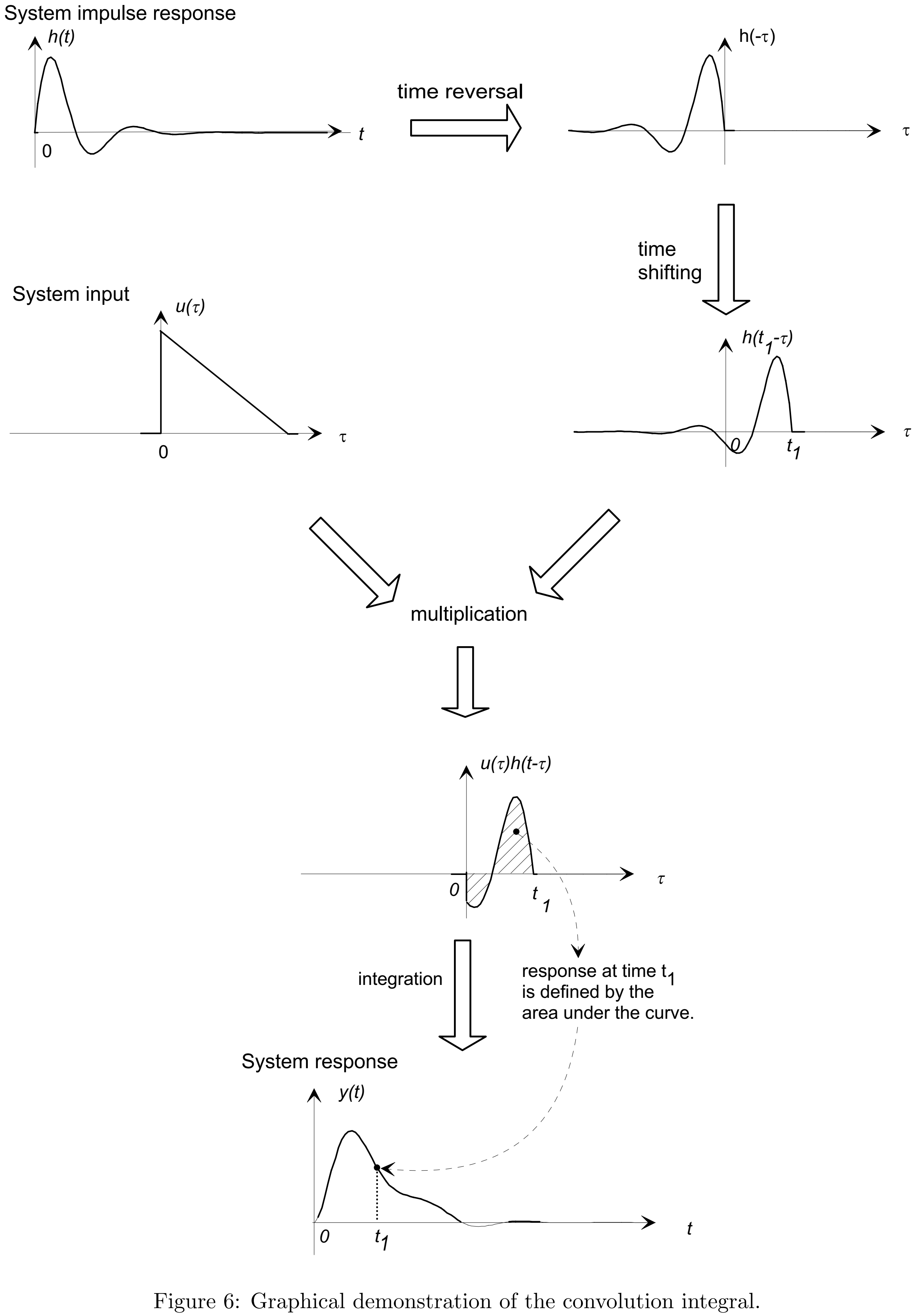

Figure 6: Graphical demonstration of the convolution integral.

Equation (18) is in the form of a linear operator, in that it transforms, or maps, an input function to an output function through a linear operation. It is a direct computational form of the system transfer operator $H\{u(t)\}$, that is:

$$y(t) = H\{u(t)\} \equiv u(t) \otimes h(t).$$

The form of the integral in Eq. (16) is difficult to interpret because it contains the term $h(t-\tau)$ in which the variable of integration has been negated. The steps implicitly involved in computing the convolution integral may be demonstrated graphically as in Fig. 6, in which the impulse response $h(\tau)$ is reflected about the origin to create $h(-\tau)$, and then shifted to the right by t to form $h(t-\tau)$. The product $u(t)h(t-\tau)$ is then evaluated and integrated to find the response. This graphical representation is useful for defining the limits necessary in the integration. For example, since for a physical system the impulse response $h(t)$ is zero for all $t < 0$, the reflected and shifted impulse response $h(t-\tau)$ will be zero for all time $\tau > t$. The upper limit in the integral is then at most t. If in addition the input $u(t)$ is time limited, that is $u(t) \equiv 0$ for $t < t_1$ and $t > t_2$, the limits are:

$$y_f(t) = \begin{cases} \displaystyle\int_{t_1}^{t} u(\tau)h(t-\tau)d\tau & \text{for } t < t_2 \\ \displaystyle\int_{t_1}^{t_2} u(\tau)h(t-\tau)d\tau & \text{for } t \geq t_2 \end{cases} \tag{19}$$

■ Example

A mass element, shown in Fig. 7 at rest on a viscous plane, is subjected to a very short unit impulsive force of duration 0.001 seconds and magnitude 1000 newtons, and is observed to respond with a velocity $v_m(t) = e^{-3t}$. Find the response of the same mass

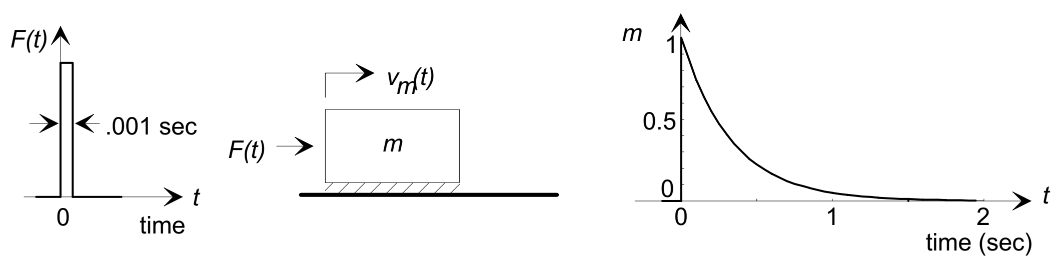

Figure 7: A sliding mass element and its impulse response.

element to a ramp in applied force $F(t) = t$ for $t > 0$.

Solution: The product of the impulsive force and its duration is unity, and because of its brief duration, the pulse may be considered to approximate an impulse. The measured response may then be taken as the system impulse response $h(t)$, and we assume that

$$h(t) = e^{-3t}. \tag{20}$$

The response to a ramp in input force, $F(t) = t$ for $t > 0$, may be found by direct substitution into the convolution integral using the assumed impulse response:

$$v(t) = \int_0^t \tau e^{-3(t-\tau)} d\tau \tag{21}$$

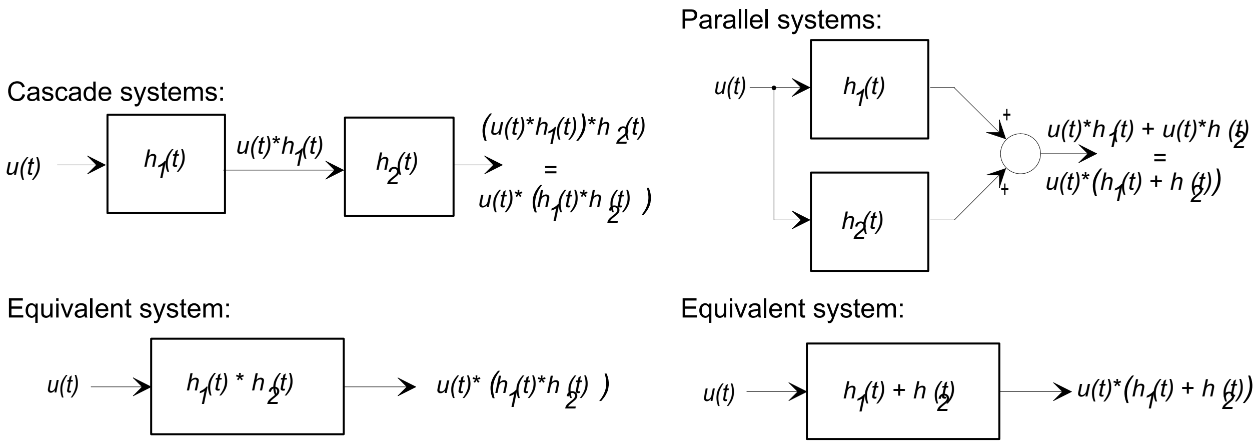

Figure 8: Impulse response of series and parallel connected systems.

$$= \quad e^{-3t} \int_0^t \tau e^{3\tau} d\tau \tag{22}$$

where the limits have been chosen because the system is causal, and the input is identically zero for all $t < 0$. Integration by parts gives the solution

$$v(t) = \frac{1}{3}t - \frac{1}{9} + \frac{1}{9}e^{-3t}. \tag{23}$$

Convolution is a linear operation and is commutative, associative and distributive, that is

$$\begin{aligned}
u(t) \otimes h(t) &= h(t) \otimes u(t) && \text{(commutative)} \\
u(t) \otimes [h_1(t) \otimes h_2(t)] &= [u(t) \otimes h_1(t)] \otimes h_2(t) && \text{(associative)} \\
u(t) \otimes [h_1(t) + h_2(t)] &= [u(t) \otimes h_1(t)] + [u(t) \otimes h_2(t)] && \text{(distributive).}
\end{aligned} \tag{24}$$

The associative property may be interpreted as an expression for the response on two systems in cascade or series, and indicates that the impulse response of two systems is $h_1(t) \otimes h_2(t)$, as shown in Fig. 8. Similarly the distributive property may be interpreted as the impulse response of two systems connected in parallel, and that the equivalent impulse response is $h_1(t) + h_2(t)$.