2.14 Fall 2004

Digital Control - Z-plane analysis

Difference Equations and Dynamics

- Consider the Discrete Transfer Function:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{K}{(z-p)} \quad \Longrightarrow \quad Y(z)(z-p) = KU(z)$$

The corresponding Difference Equation is:

$$y_{i+1} - py_i = Ku_i \Rightarrow y_{i+1} = py_i + Ku_i$$

next output = p*current output + K* current input

Step Dynamics

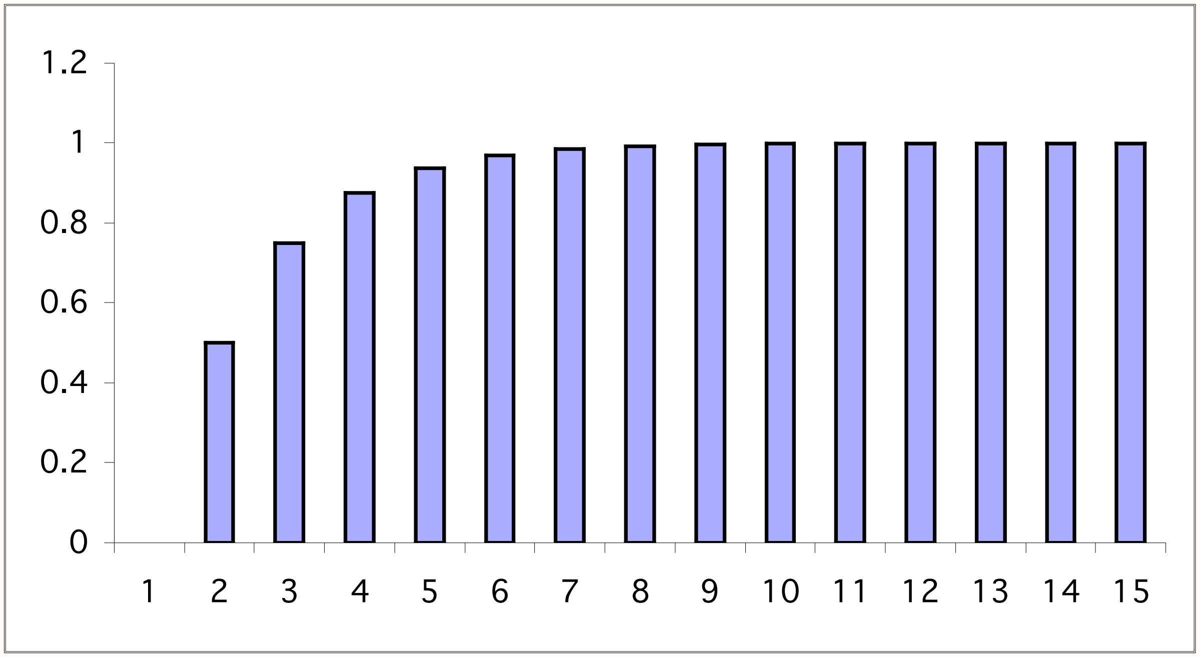
$$y_{i+1} = py_i + pu_i$$

for p=0.5 and K=0.5 and u=unit step:

$$\Rightarrow y_{i+1} = 0.5(y_i + u_i)$$

I	u_i	y_i
1	1	0.000
2	1	0.500
3	1	0.750
4	1	0.875
5	1	0.938
6	1	0.969
7	1	0.984
8	1	0.992
9	1	0.996
10	1	0.998
11	1	0.999
12	1	1.000
13	1	1.000

Step Response

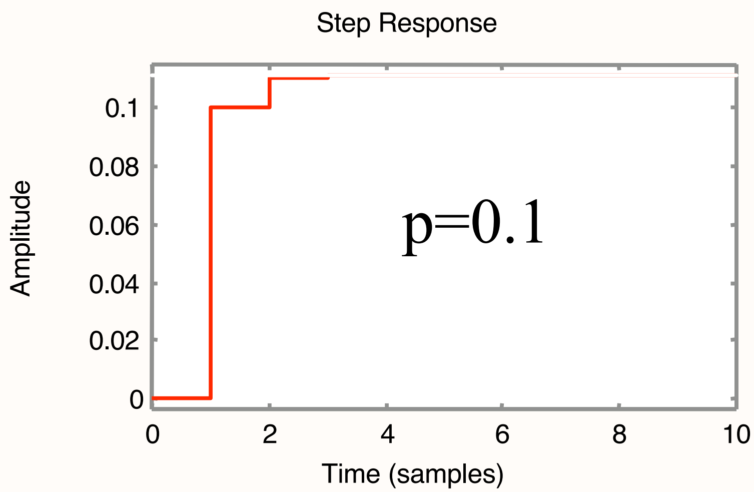


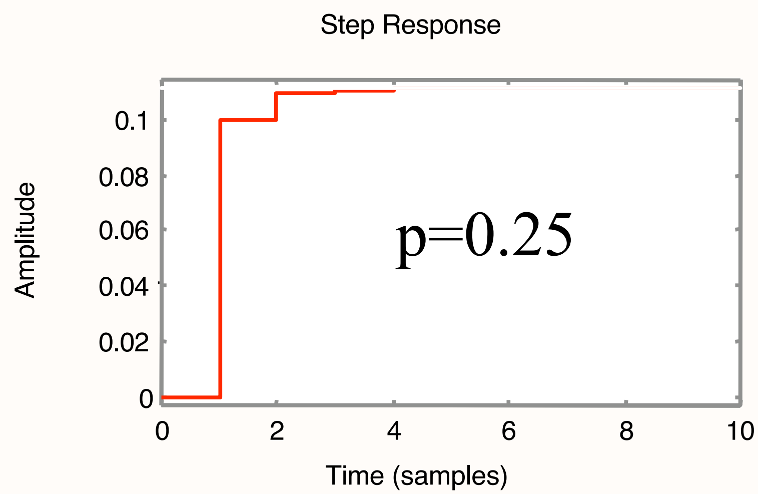
$$G(z) = \frac{Y(z)}{U(z)} = \frac{K}{(z-p)} = \frac{0.5}{z-0.5}$$

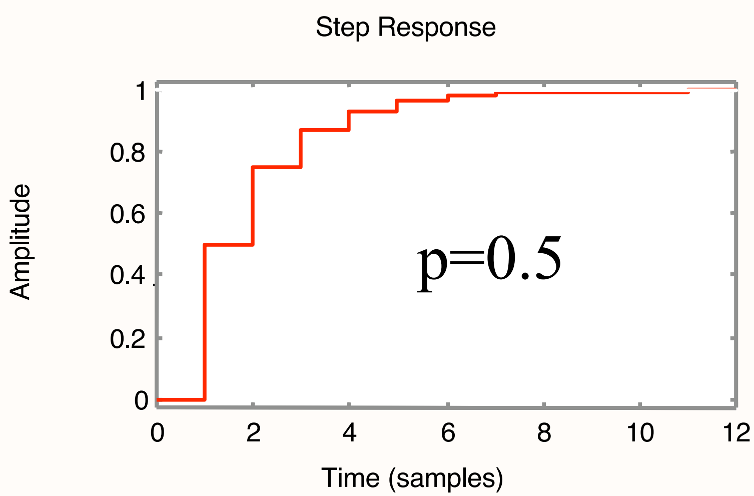
First Order

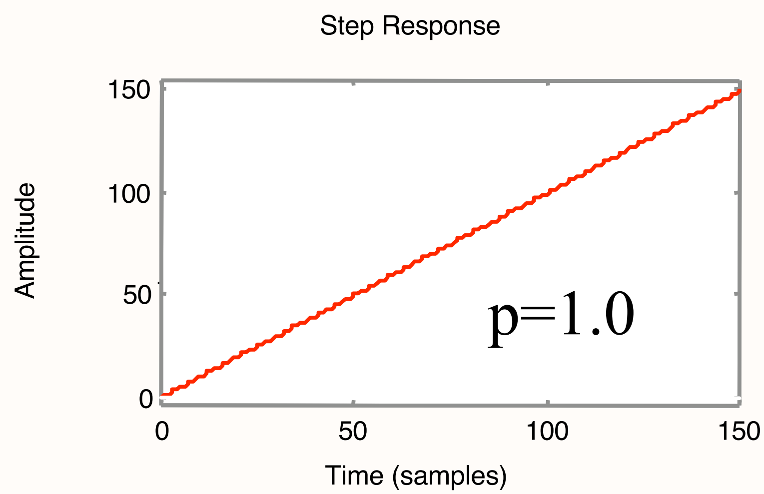
Transfer Function!

Effect of Root *p*

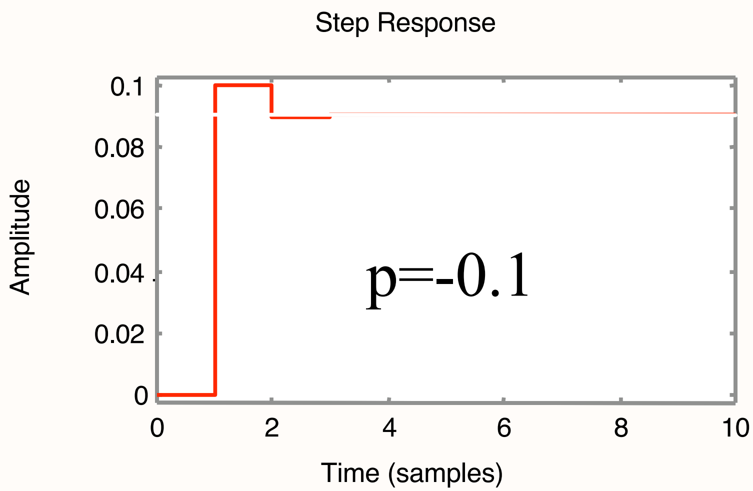


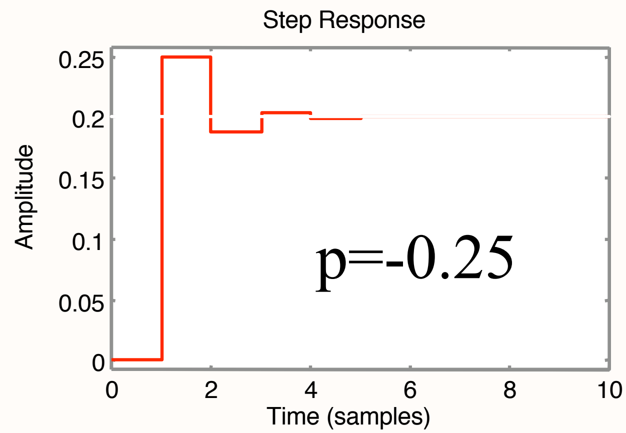


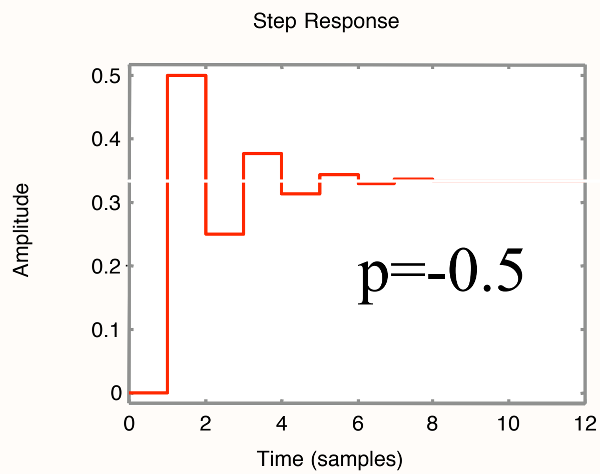


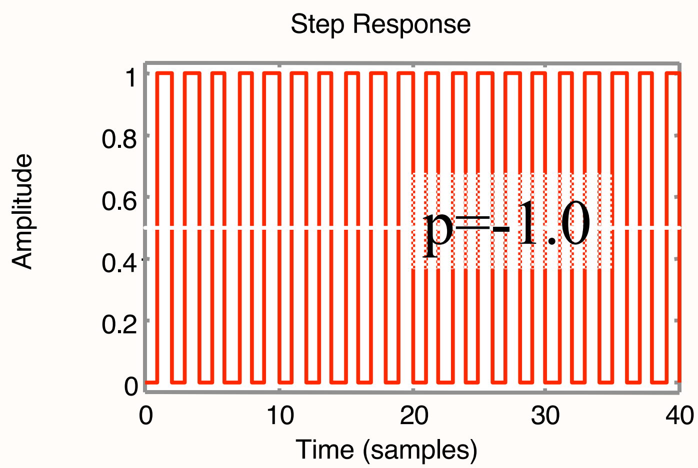


Effect of Root *p*







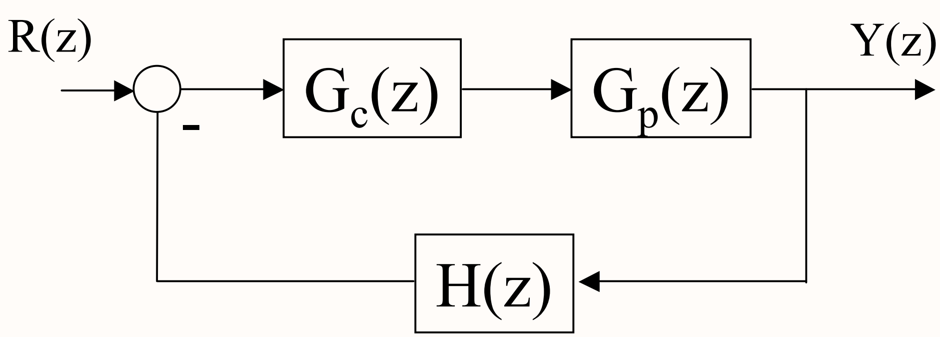


What's Next?

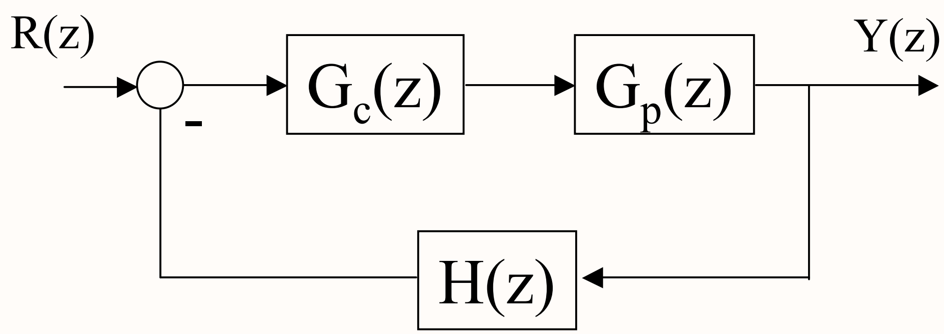
- Z- domain modeling of control system
- Root locus in z-domain (it's the same!)
- Cycle to Cycle Stability Limits

Analysis of Dynamics of Discrete Feedback Systems

- Given the Discrete Transfer Function For the Process $G_p(z)$
- What are the Dynamics of the resulting Closed Loop System?:



Closed-Loop Dynamics



$$\frac{Y(z)}{R(z)} = T(z) = \frac{G_c G_p}{1 + G_c G_p H}$$

and the characteristic equation is:

$$1 + G_c(z) G_p(z) H(z) = 0$$

Z-Domain Root Locus

In general the CE:

$$1 + G_c G_p H(z) = 0$$

will be a polynomial in z:

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots a_1 z + a_0 = 0$$

and the roots of this polynomial will define the dynamics of our system

S-Z Plane Mapping

Recall the Definition: $z = e^{sT}$

And that $s = \sigma + j\omega$

then z in polar coordinates is given by:

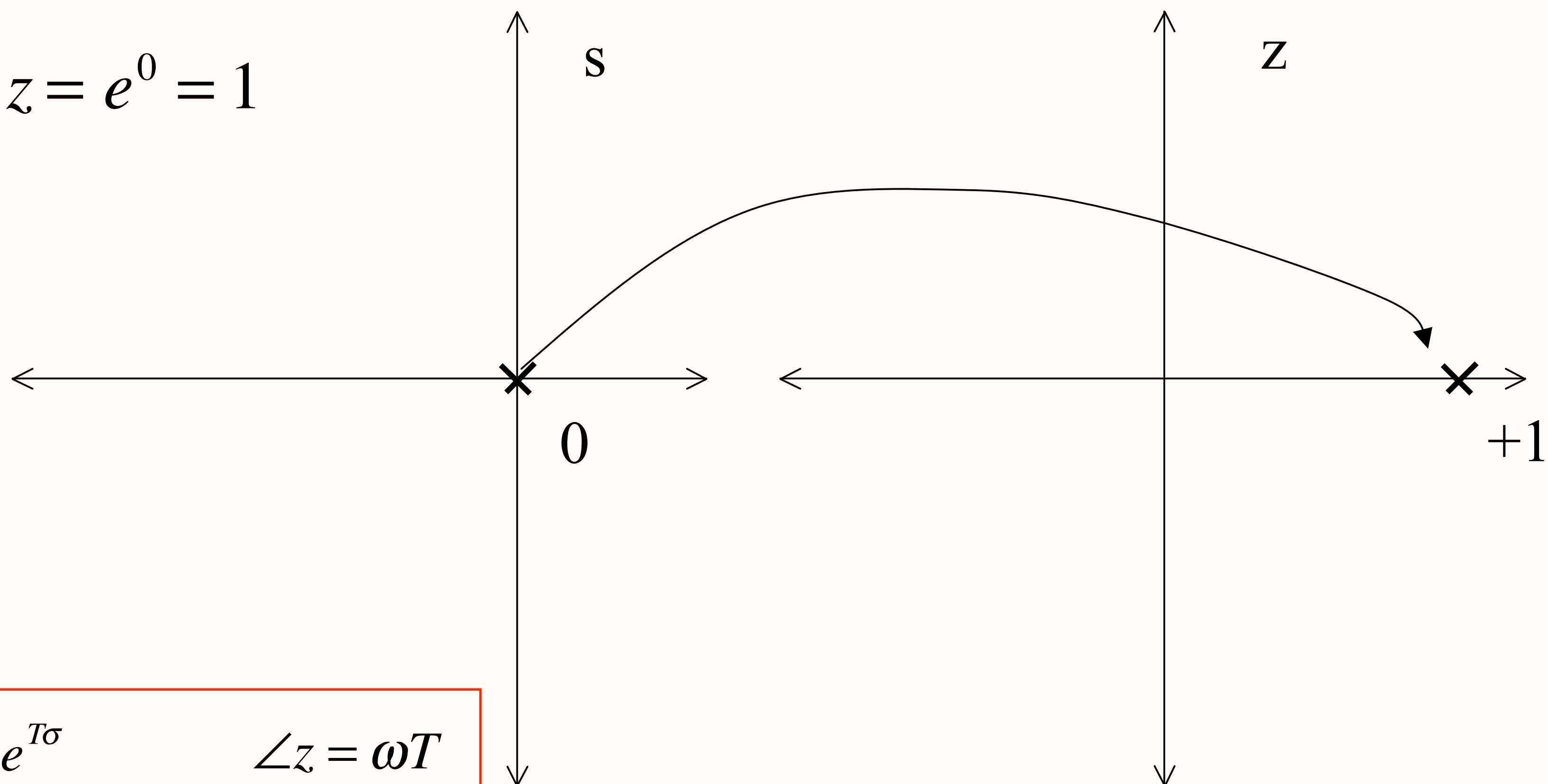
$$e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}$$

$$|z| = e^{T\sigma} \qquad \angle z = \omega T$$

Mapping

$$s = 0$$

$$z = e^0 = 1$$

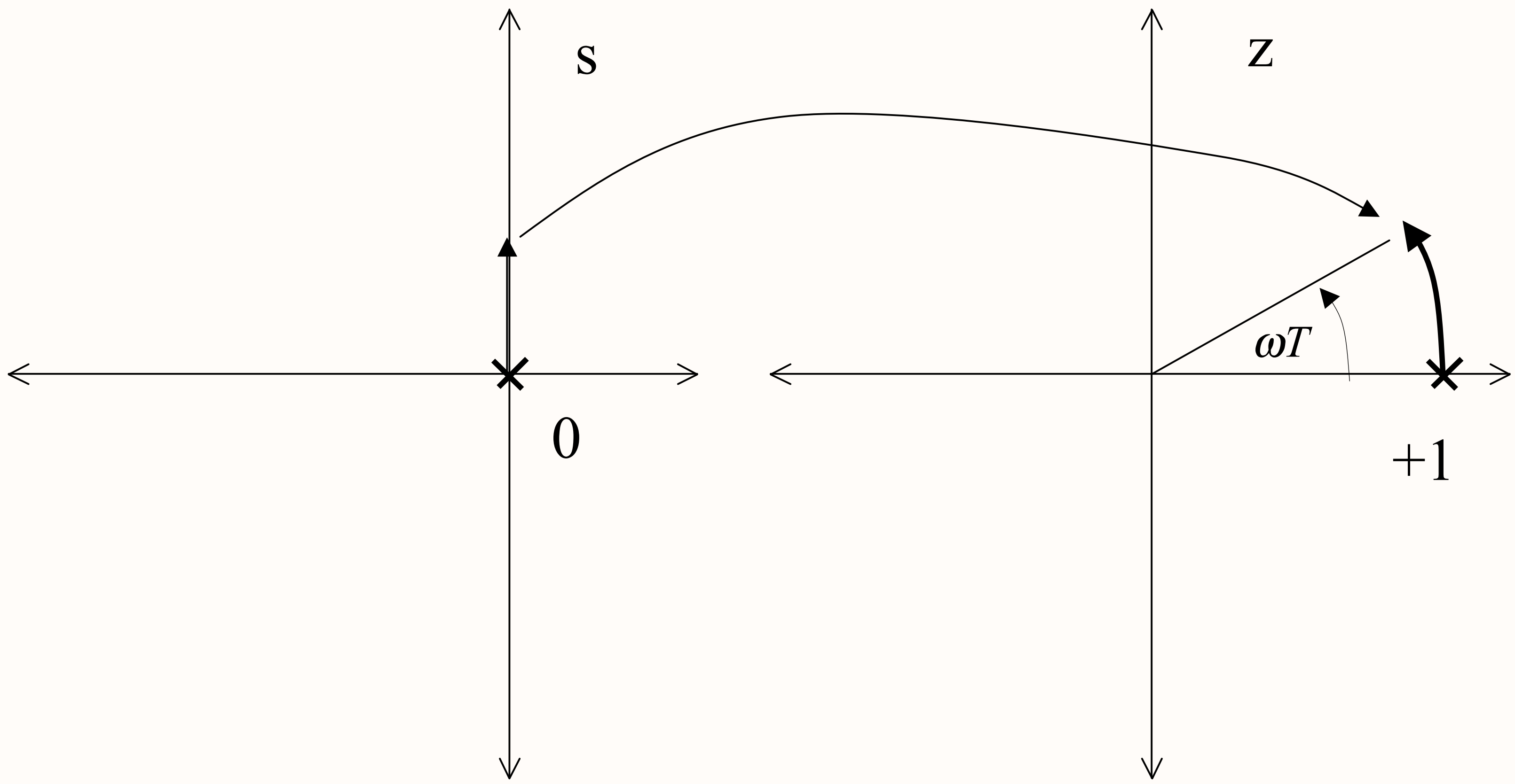


$$|z| = e^{T\sigma} \qquad \angle z = \omega T$$

Mapping $j\omega$ Axis

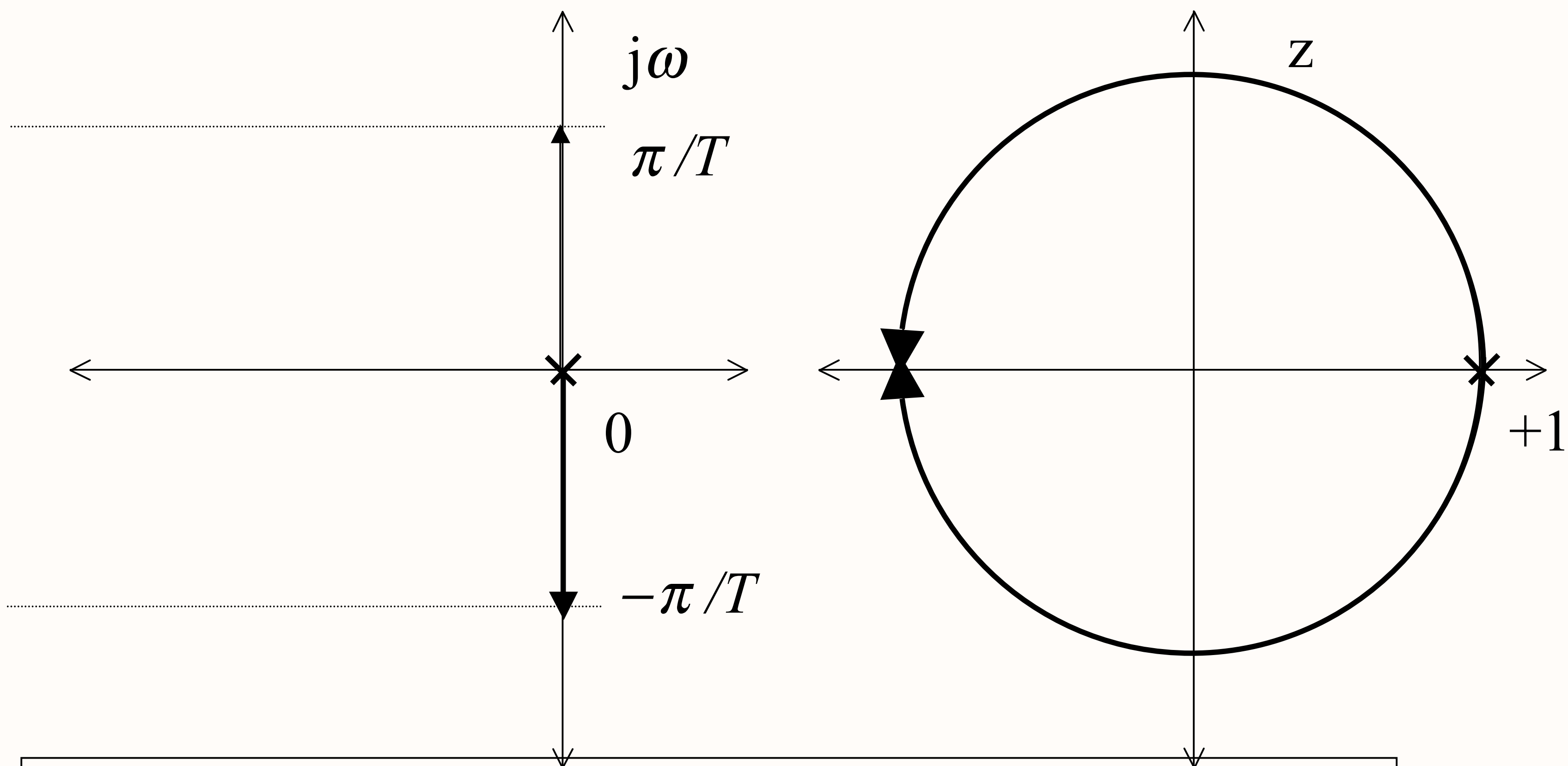
$s = +j\omega$

$z = e^{j\omega T}$ = unit length vector at angle ωT



Mapping $j\omega$ Axis

Note: when $\omega T = \pi$, we have reached -1 on z-plane

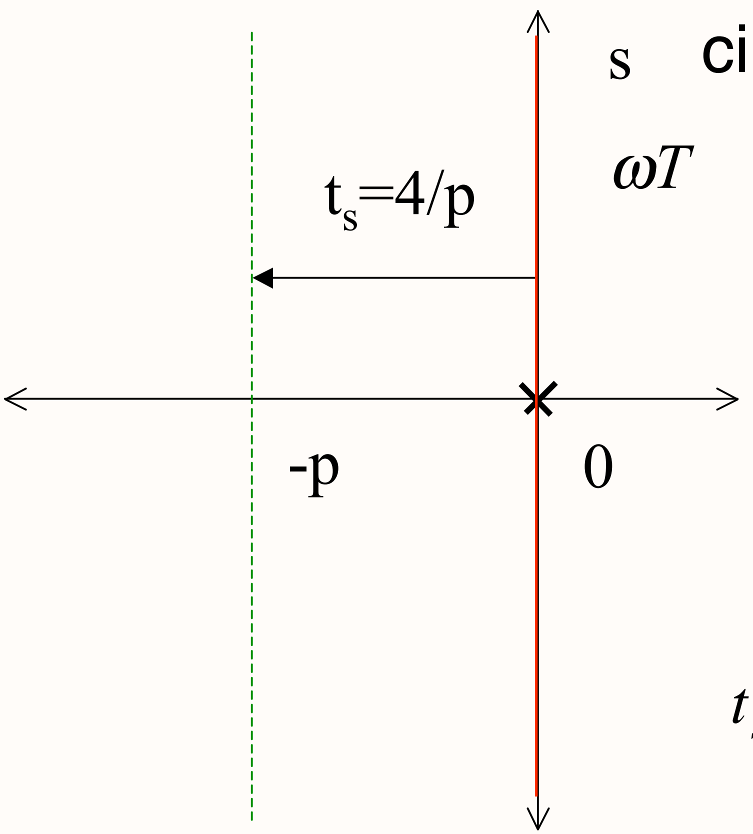


And when $\omega T = -\pi$, we have come around the other way

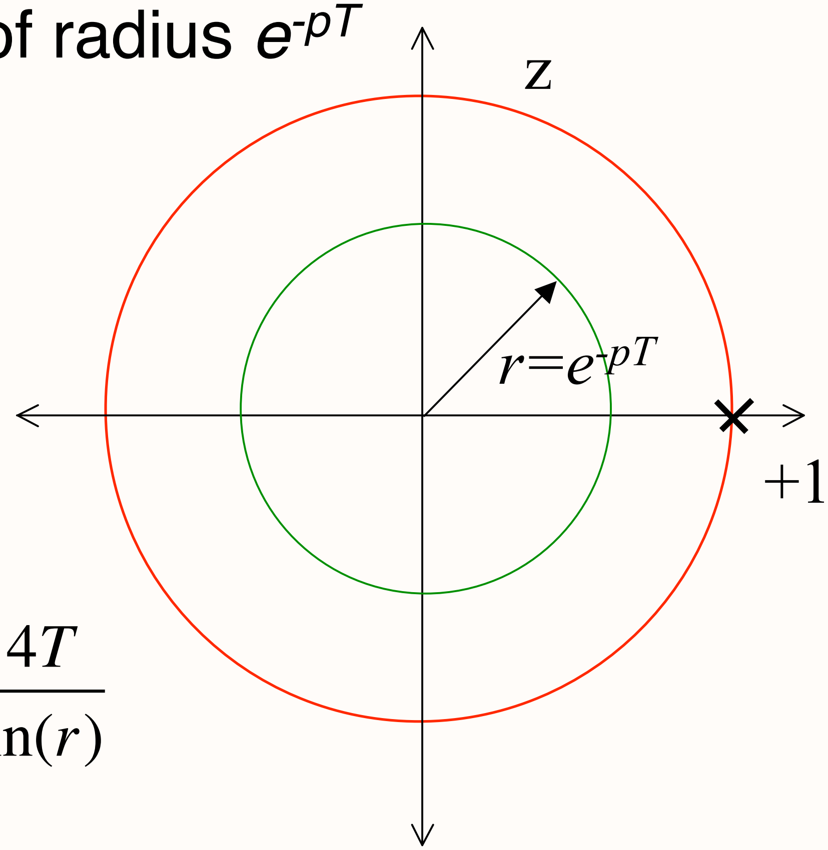
Settling Time Mapping

S-plane lines of constant settling: $s = -p + j\omega$

$$z = e^{-pT + j\omega T} = e^{-pT} e^{j\omega T}$$



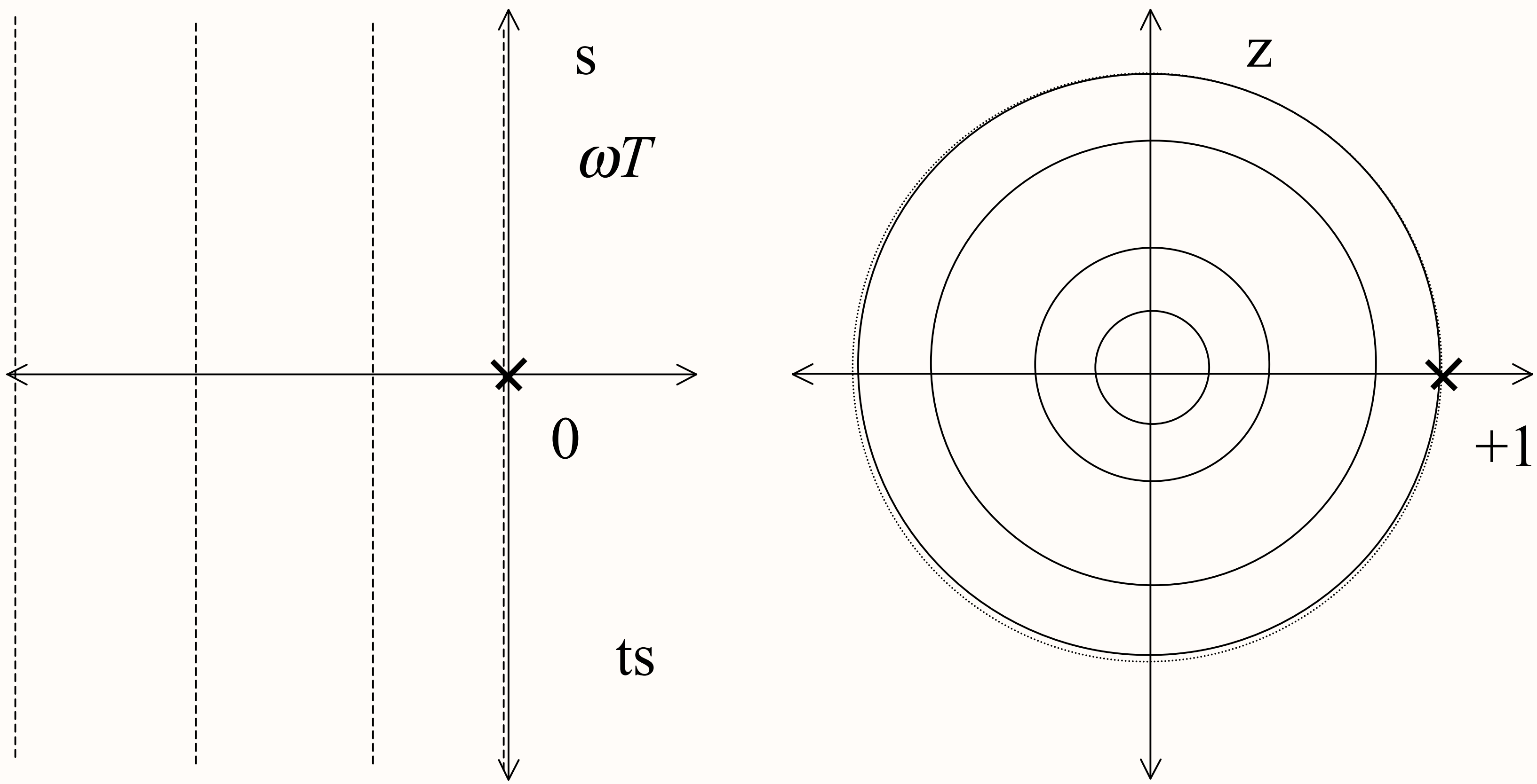
circle of radius e^{-pT}



$$t_s \approx -\frac{4T}{\ln(r)}$$

Settling Time Mapping

circle of radius e^{-pT}

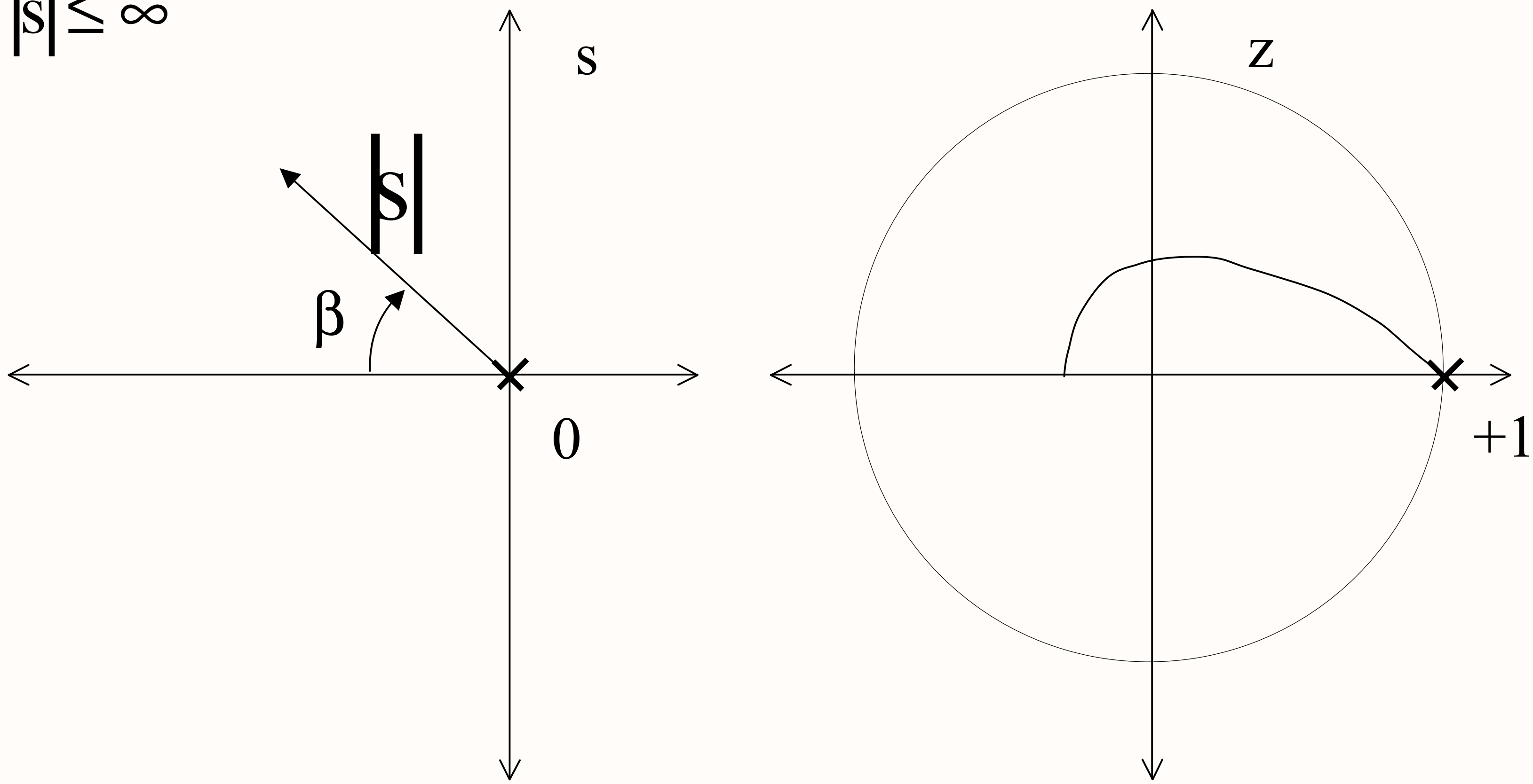


CLOSER TO Z=0 = FASTER SETTLING

Damping Ratio Mapping

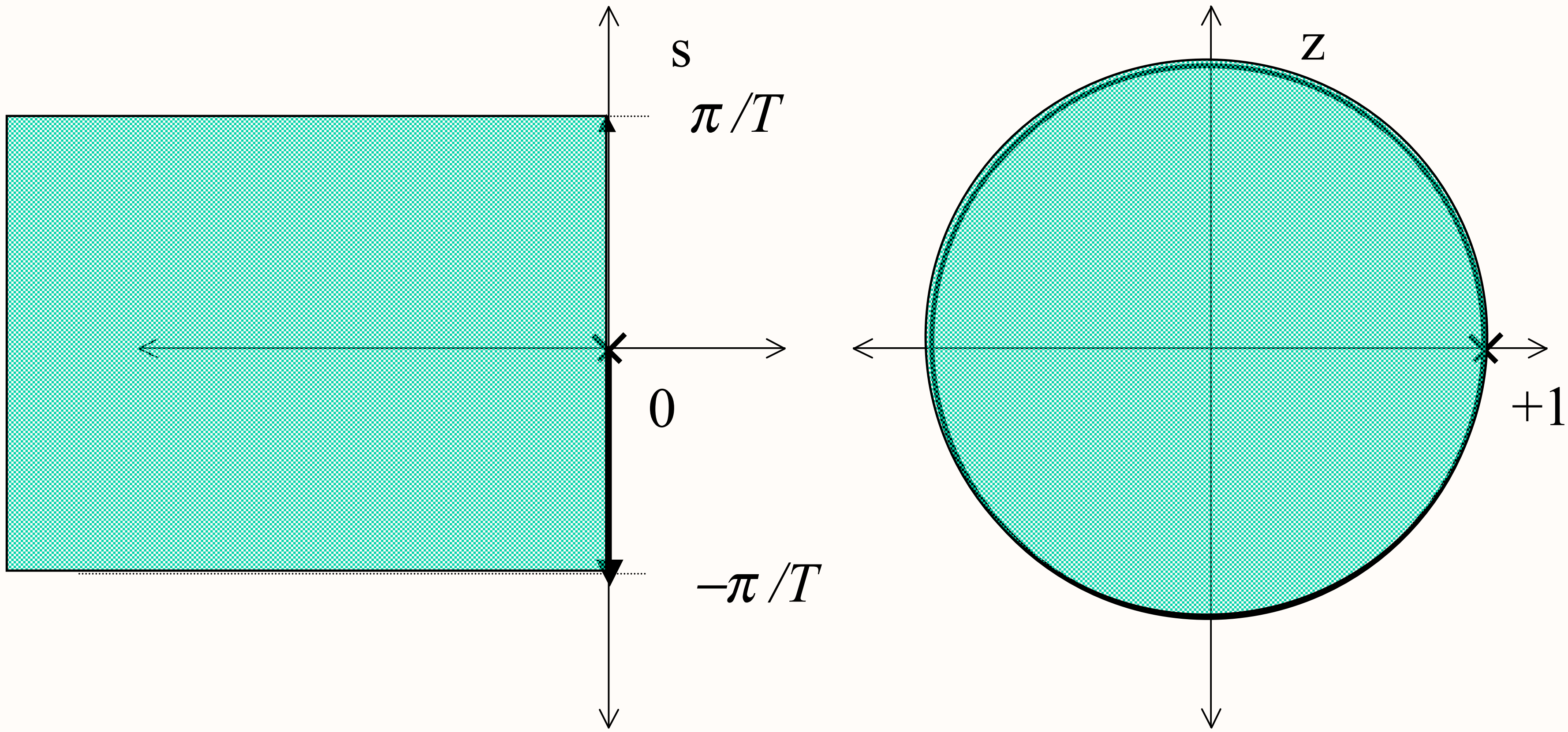
$$\angle s = \beta = \text{constant}$$

$$0 \leq |s| \leq \infty$$



Stability?

jw axis maps to the “unit circle”

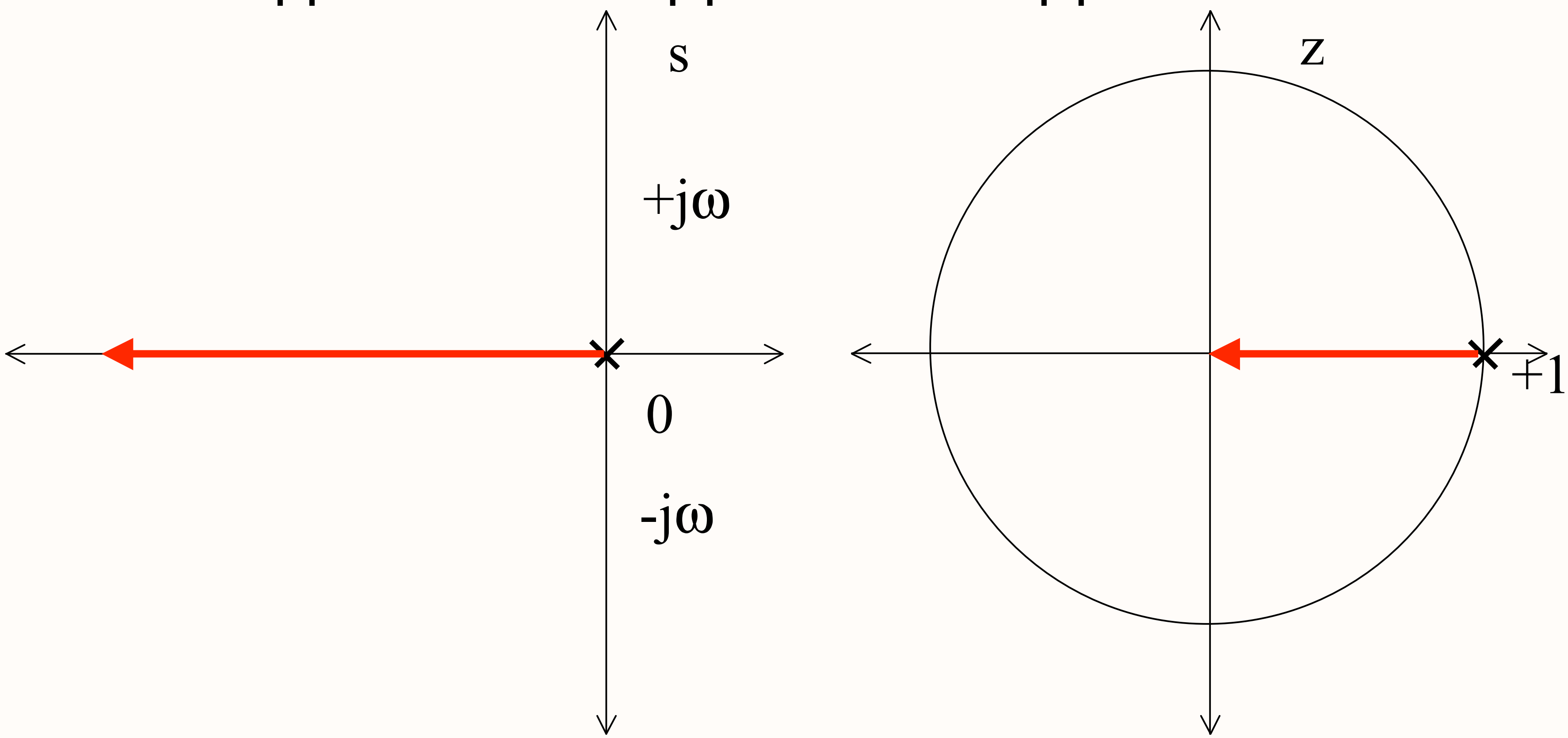


roots inside the unit circle are stable, outside are unstable

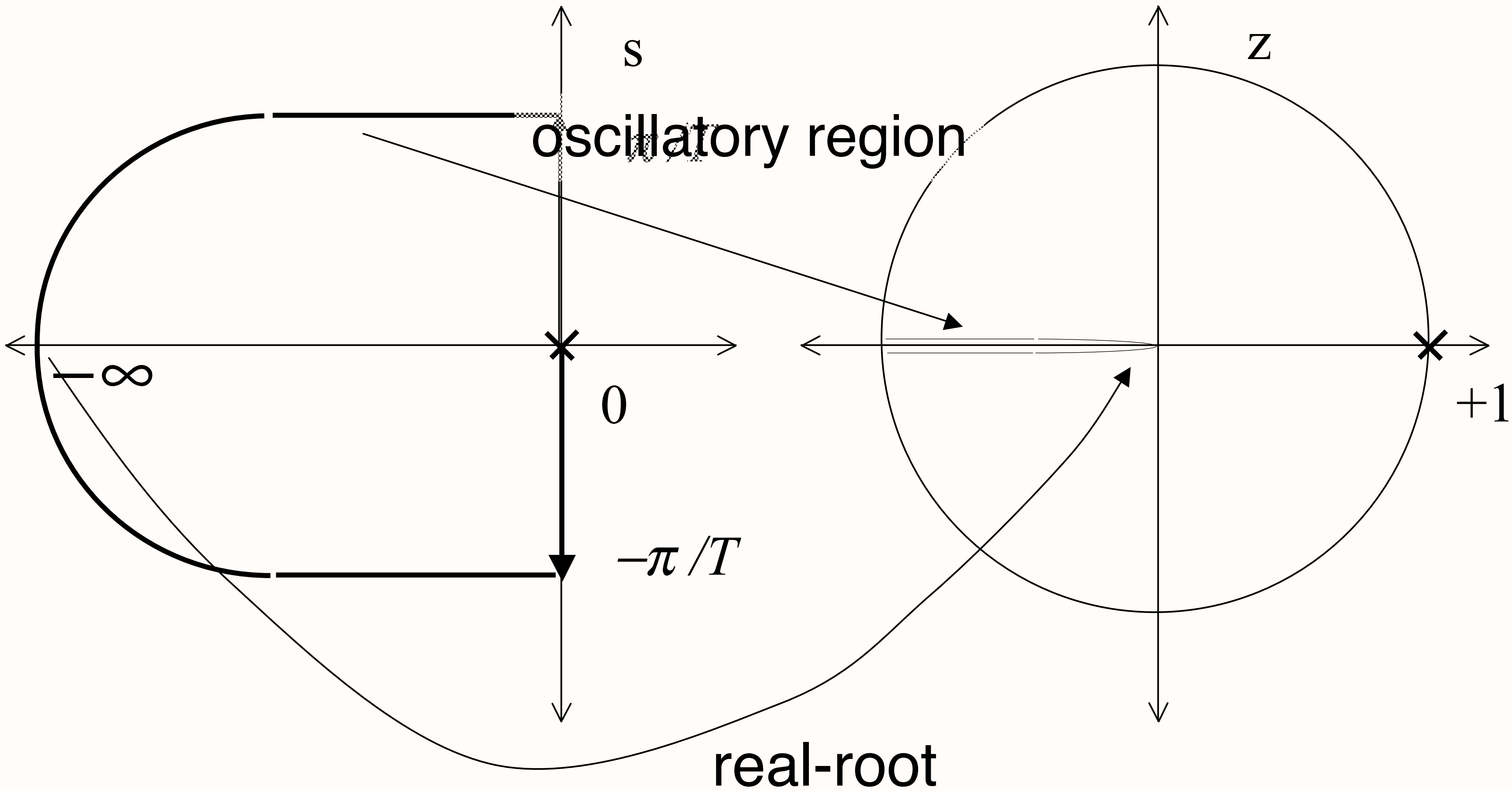
Mapping: Negative Real Axis

$$\angle z = \omega T = 0$$

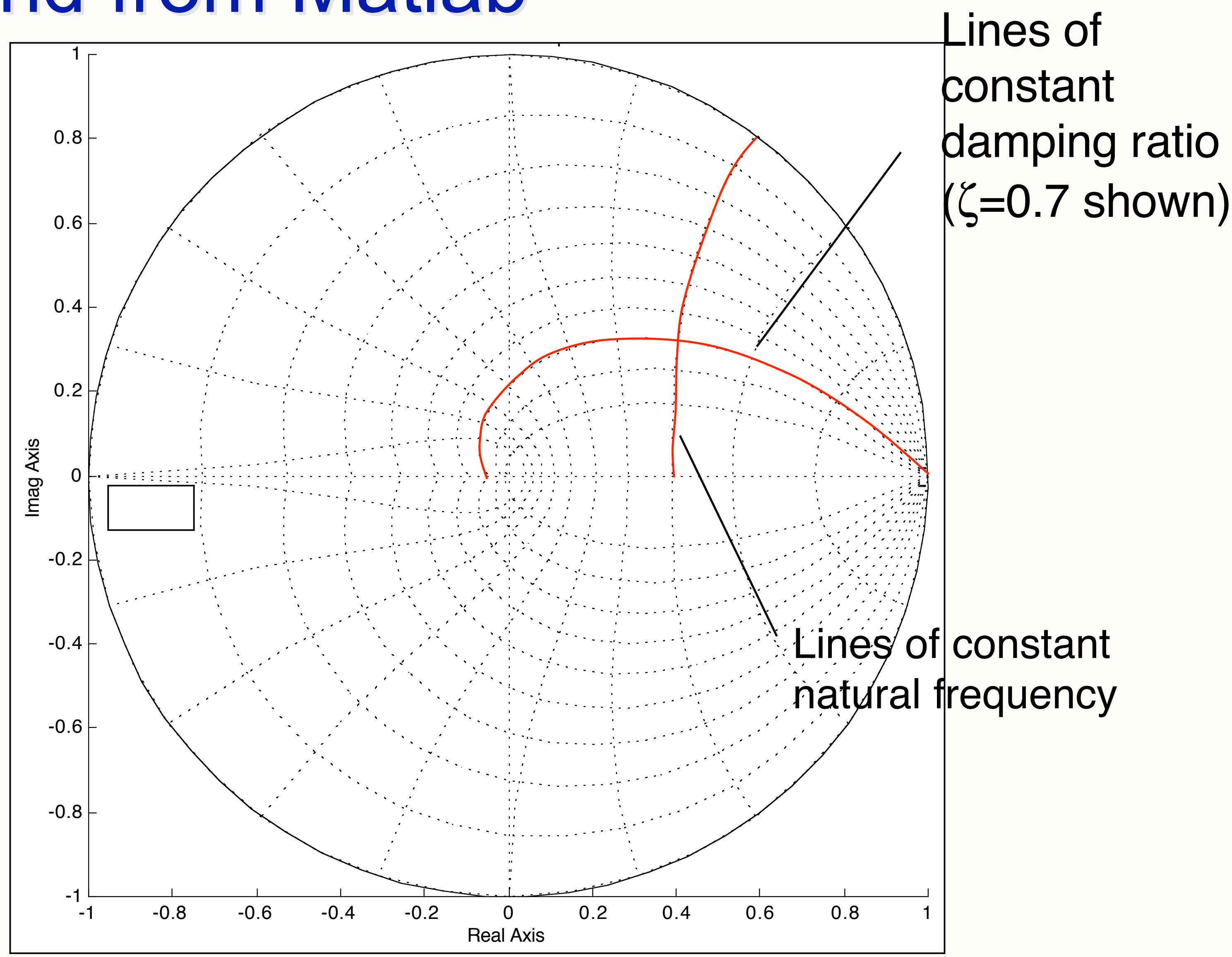
$$0 \leq |s| \leq -\infty \Rightarrow e^{0} \leq |z| \leq e^{-\infty} \Rightarrow 1 \leq |z| \leq 0$$



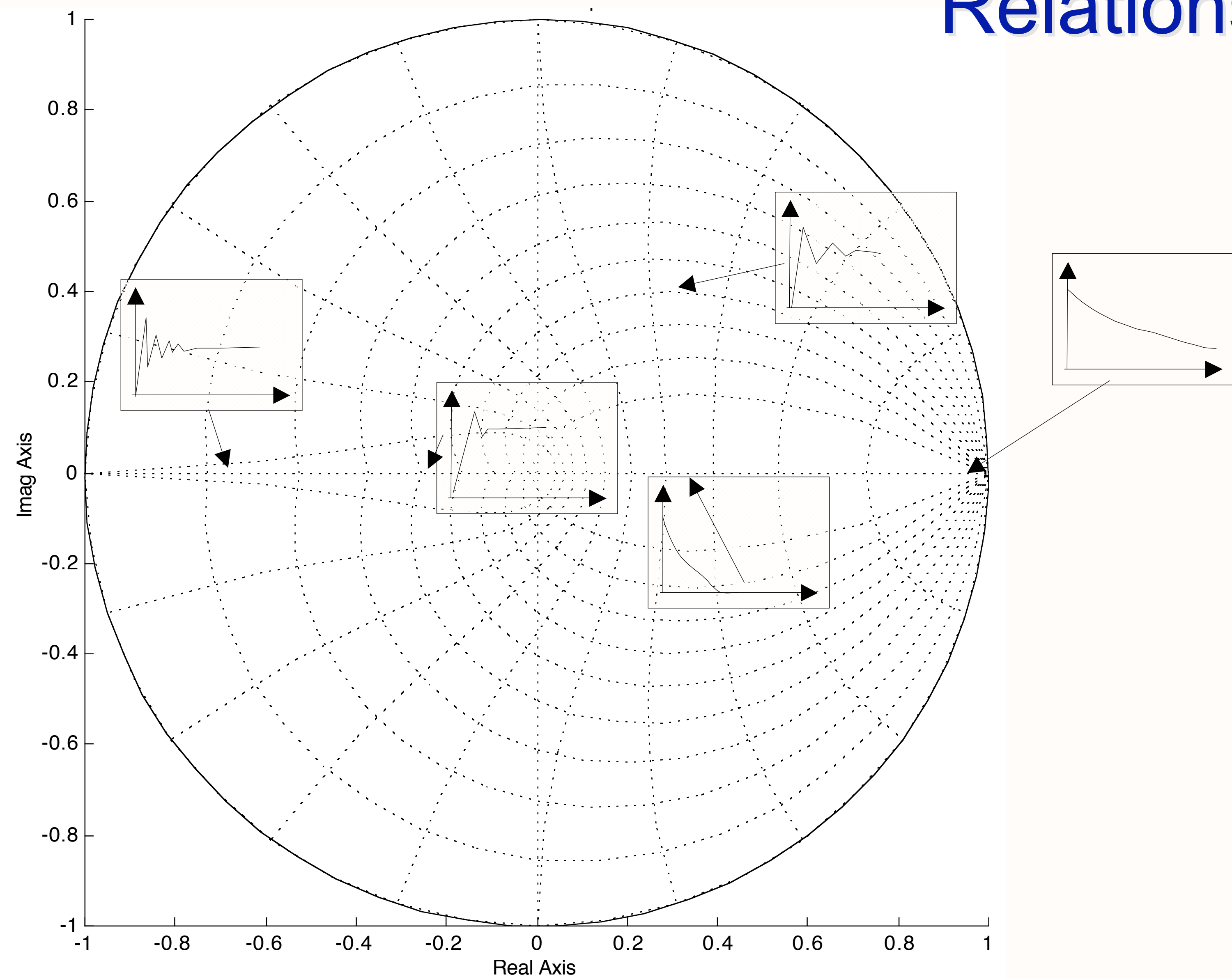
Mapping: What's left



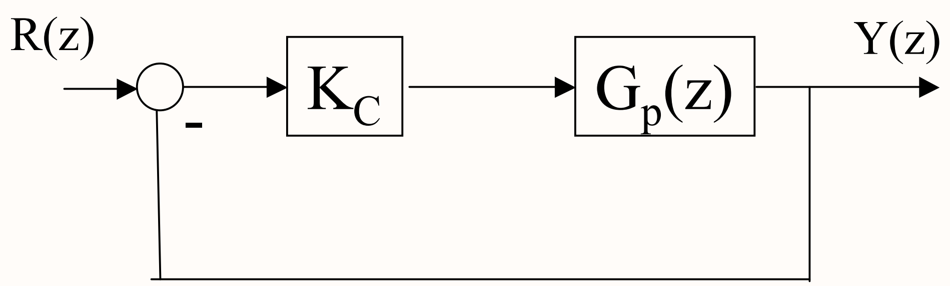
zgrid from Matlab



Z-Plane - Response Relationship



Simple First Order Control System



$$G_p(z) = \frac{K}{(z-p)} \qquad \text{open loop root} = +p$$

Root Locus

ROOTS OF $1+K_C G_{-P}(z) = 0$?

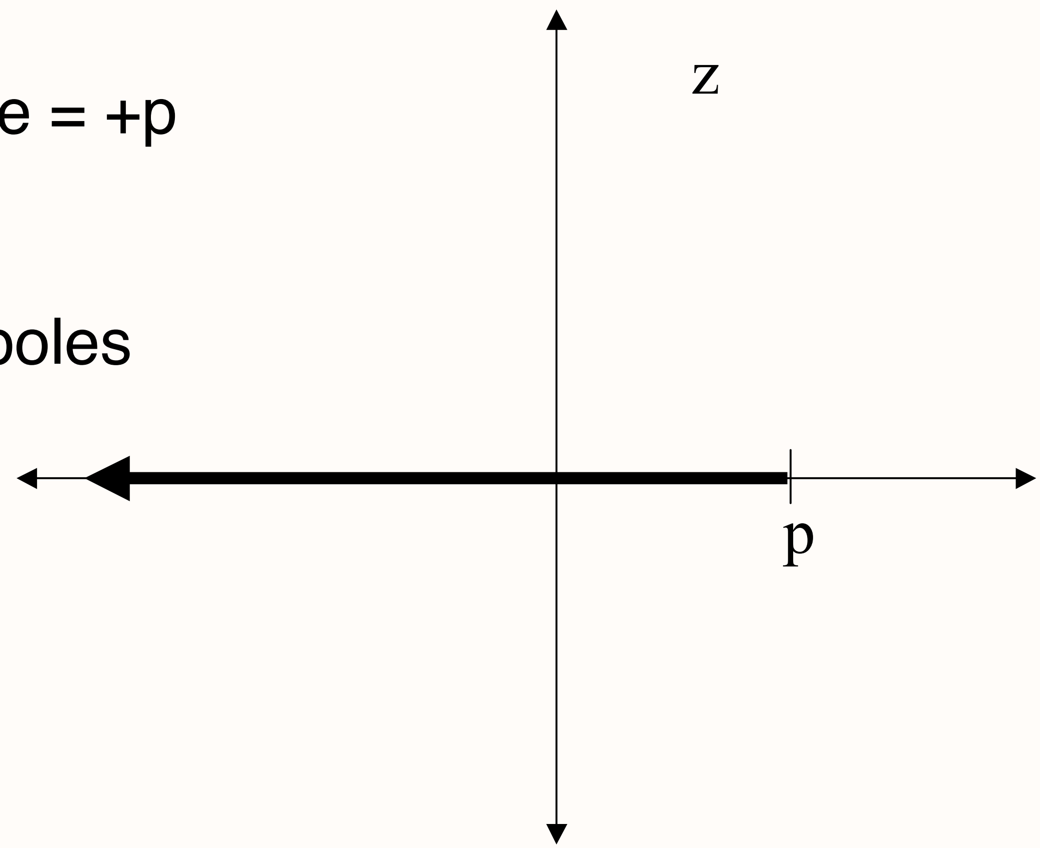
Same rules as before:

Starts at open loop pole = +p

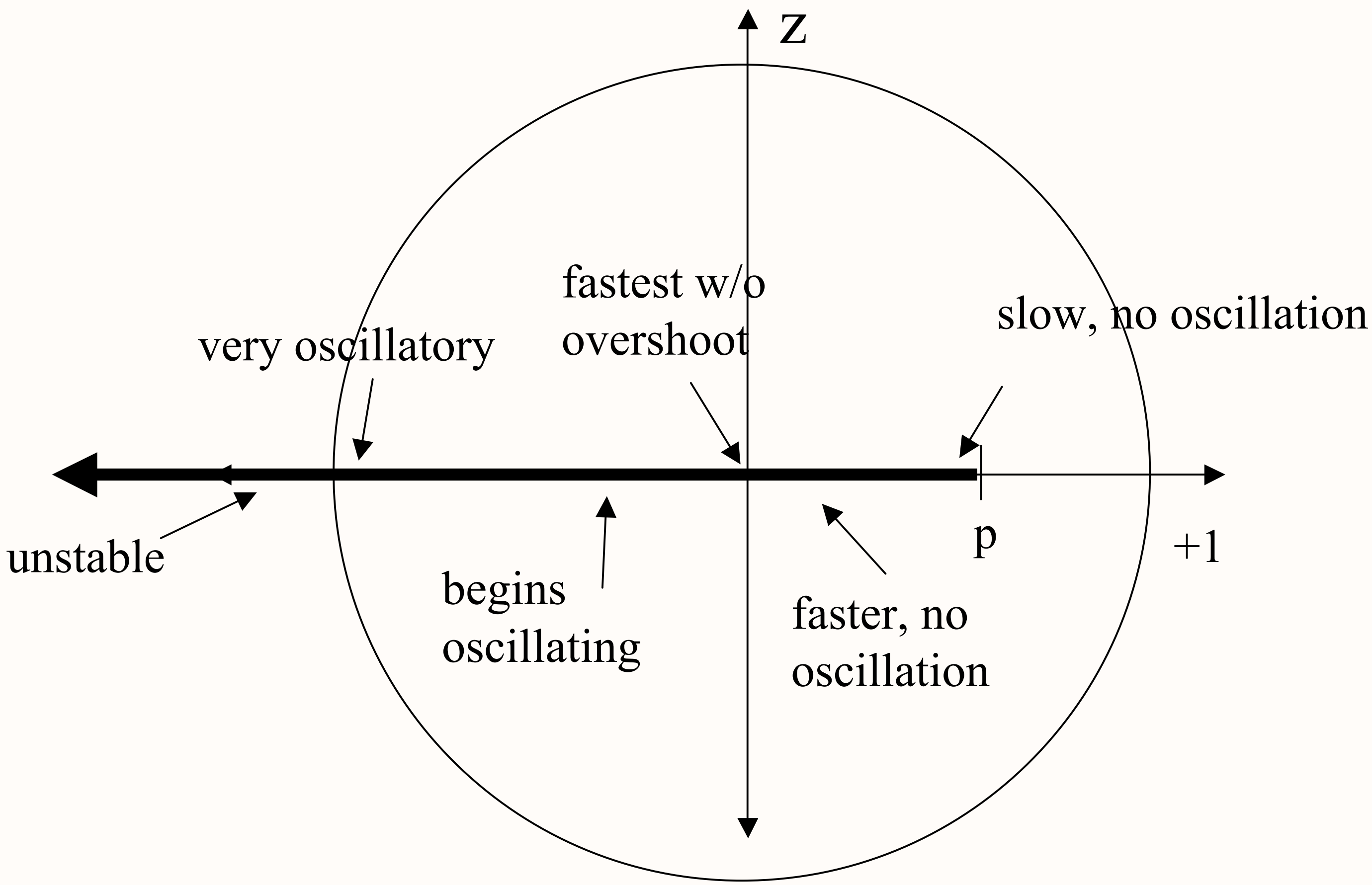
Real axis part to left of

odd number of poles

Ends at zero at infinity



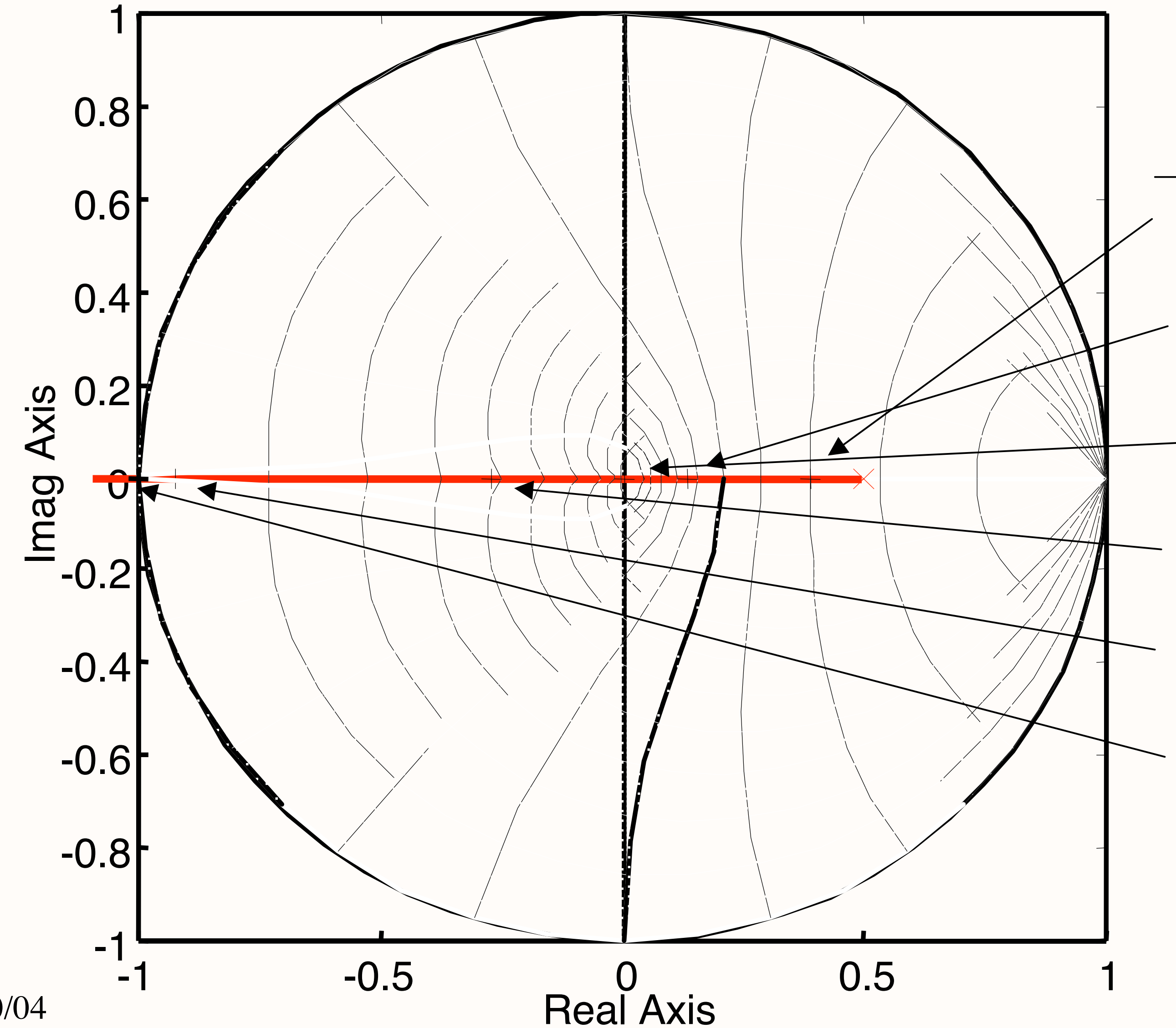
Root Locus- Interpretation



Now for

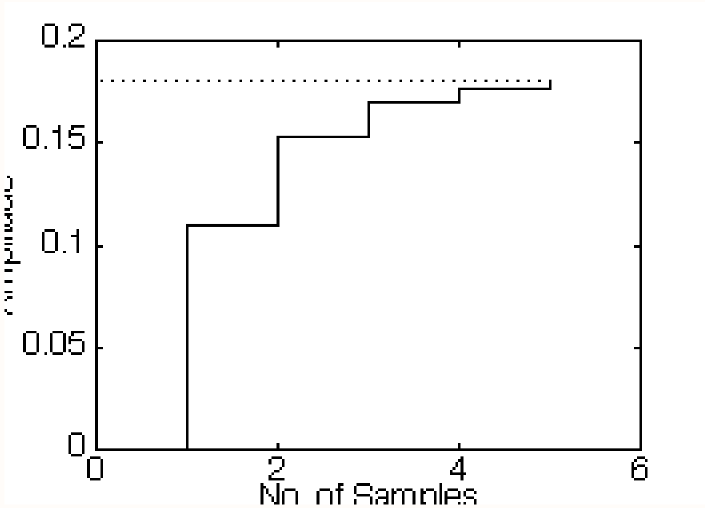
$$G(z) = \frac{K}{(z-p)} = \frac{0.5}{z-0.5}$$

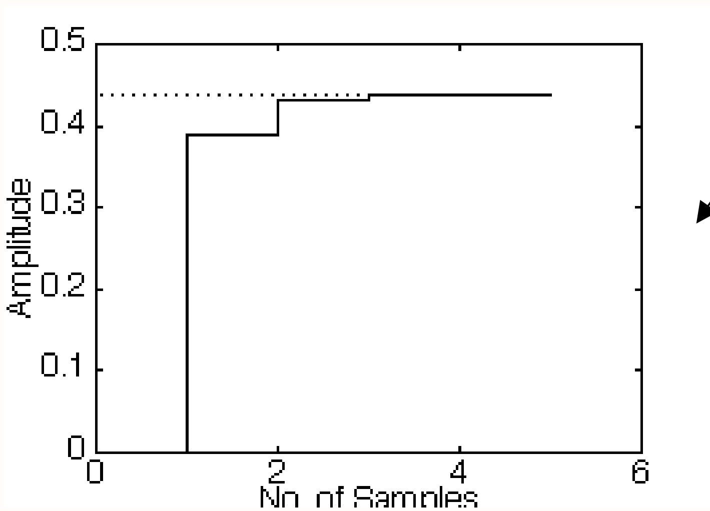
K=0.5
p=0.5



k	r
0.22	0.39
0.73	0.13
1	0
1.5	-0.27
2.8	-0.92
3	-1

Response?





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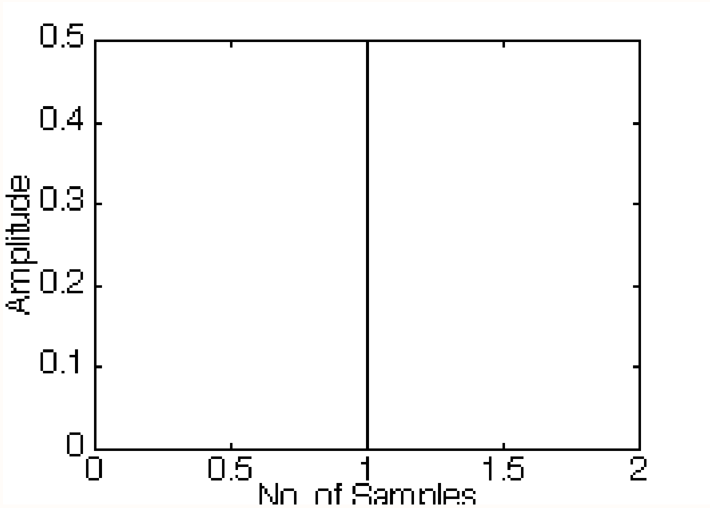

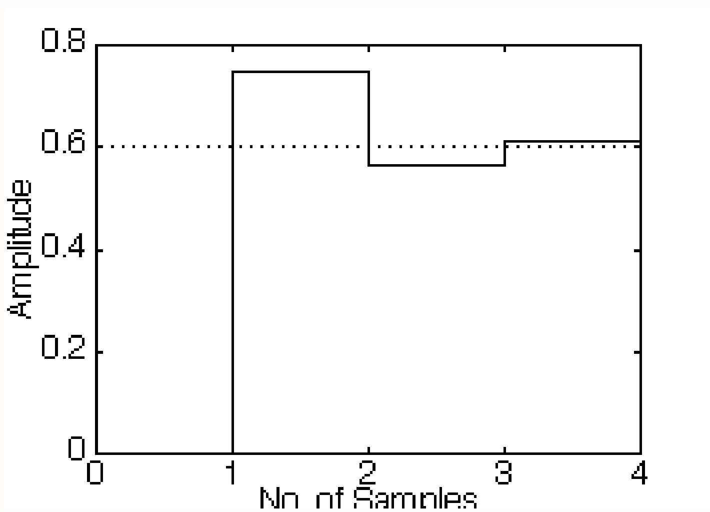

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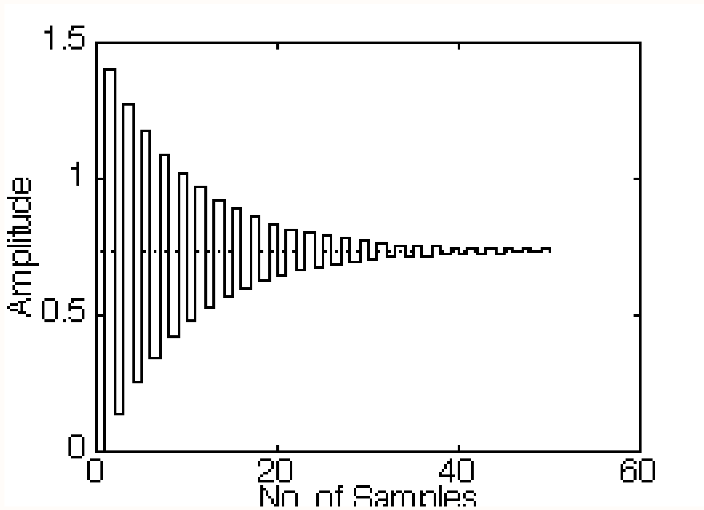

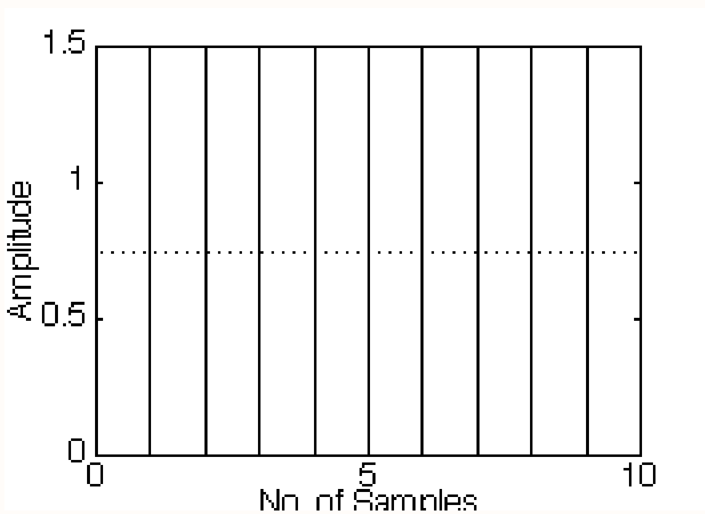

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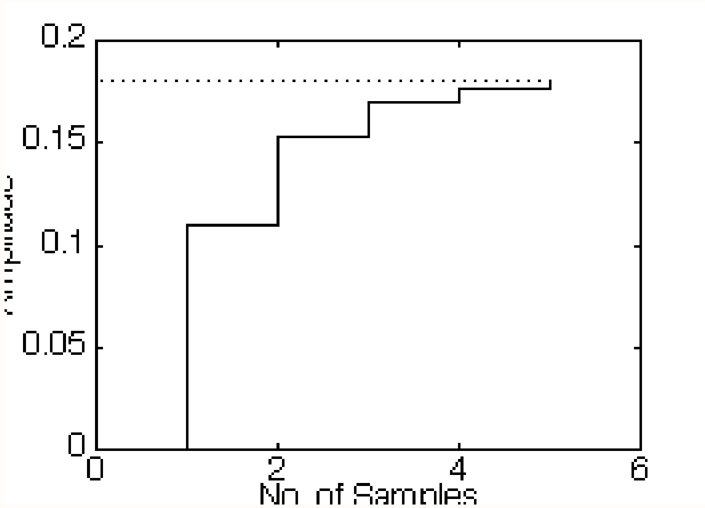

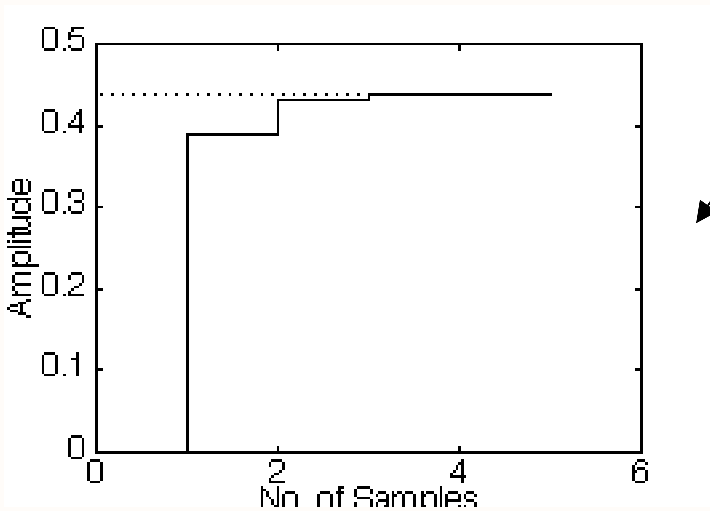

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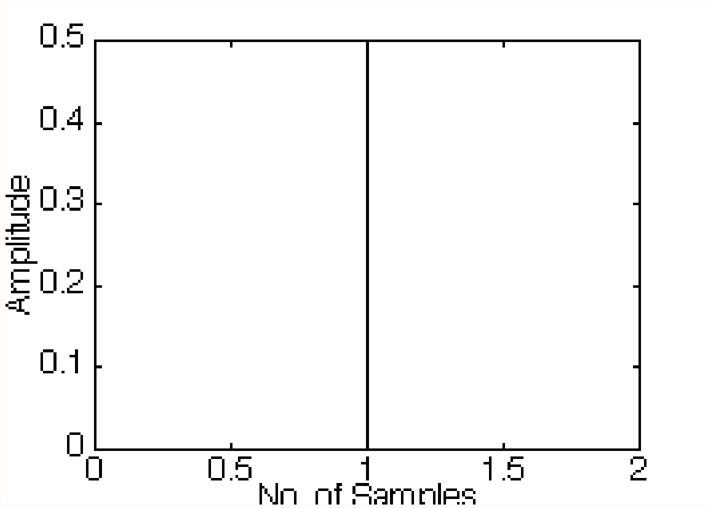

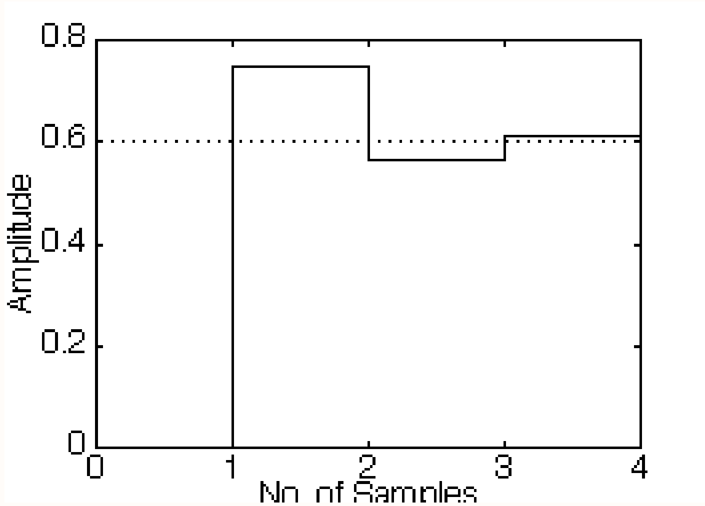

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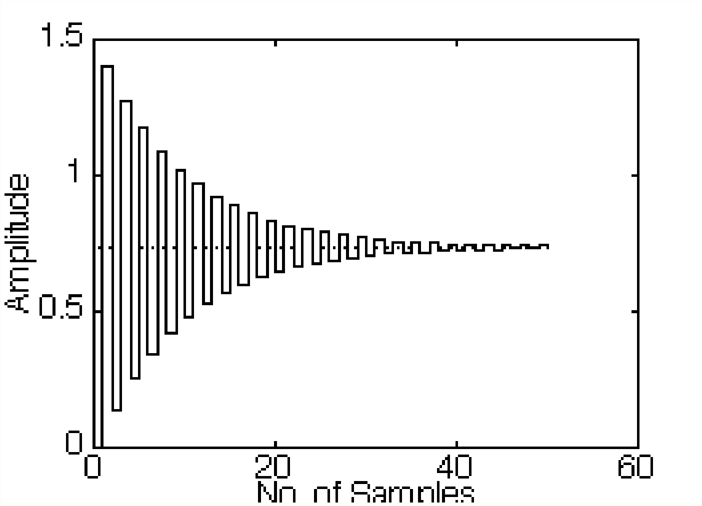

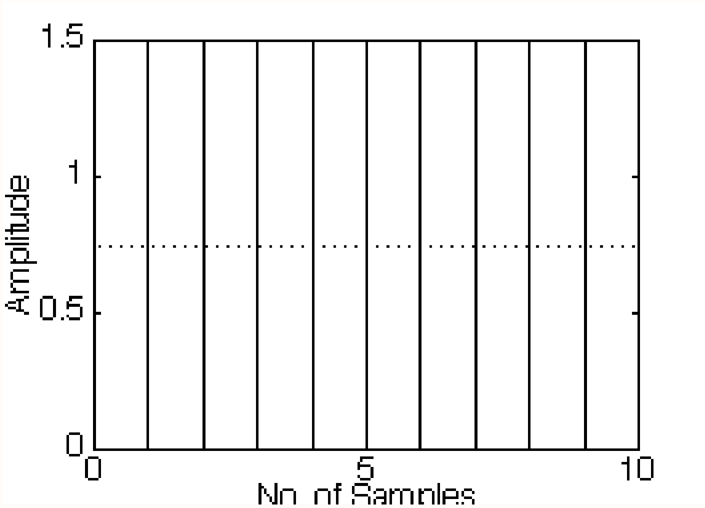

Steady State Unit Step Error?

For s-domain $y(\infty) = \lim_{s \to 0} s\, G(s)\, \frac{1}{s}$

For z-domain $y(\infty) = \lim_{z \to 1} (z-1)\, G(z)\, \frac{z}{z-1}$

unit step

Our closed-loop TF = $T(z) = \dfrac{\dfrac{K_c K}{z-p}}{1 + \dfrac{K_c K}{z-p}} = \dfrac{K_c K}{z - p + K_c K}$

Steady State Error?

$$\lim_{z \to 1}(z-1)\ T(z)\ \frac{z}{z-1} = (z-1)\ \frac{K_c K}{z-p+K_c K}\ \frac{z}{(z-1)}$$

$$= \frac{K_c K}{1-p+K_c K}$$

K_c	p	$y(\infty)$
0.22	0.39	0.18
0.73	0.13	0.44
1	0	0.5
1.5	-0.27	0.6
2.8	-0.92	0.74
3	-1	0.75

K=0.5
p=0.5

Steady State Error is Minimized
as K increases