P.S. #3 SOLUTIONS

Problem 1

Show: $y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$ $\overset{\mathcal{L}}{\longleftrightarrow}$ $Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

① rewrite $y(t)$ using Euler identity:

$$y(t) = \left[\frac{\omega_n}{\sqrt{1-\zeta^2}}\right] \frac{1}{2j} \left[e^{-\zeta\omega_n t}\right] \cdot \left[e^{j\omega_d t} - e^{-j\omega_d t}\right]$$

$$= \frac{1}{2j} \cdot \left[\frac{\omega_n}{\sqrt{1-\zeta^2}}\right] \cdot \left[e^{(-\zeta\omega_n + j\omega_d)t} - e^{(-\zeta\omega_n - j\omega_d)t}\right]$$

$$y(t) = \frac{1}{2j}\left[K\right] \cdot \left[e^{at} - e^{bt}\right]$$

Define: $K = \frac{\omega_n}{\sqrt{1-\zeta^2}}$

$a = -\zeta\omega_n + j\omega_d$

$b = -\zeta\omega_n - j\omega_d = a^*$

② Take Laplace xform:

$$Y(s) = \frac{K}{2j}\left(\frac{1}{s-a}\right) - \frac{K}{2j}\left(\frac{1}{s-b}\right)$$

$$= \frac{K}{2j}\left[\frac{(s-b)-(s-a)}{(s-a)(s-b)}\right]$$

$$= \frac{K}{2j}\left[\frac{a-b}{s^2-(a+b)s+ab}\right]$$

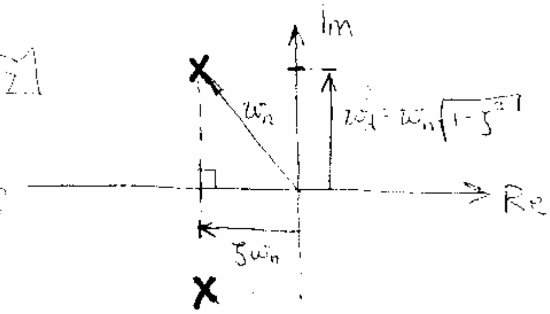
Then
$a-b = 2j\omega_d$
$a+b = -2\zeta\omega_n$
$ab = (\zeta\omega_n)^2 + (\omega_d)^2$

$$= \frac{\omega_n}{2j\sqrt{1-\zeta^2}}\left[\frac{2j\omega_d}{s^2+2\zeta\omega_n s + (\zeta\omega_n)^2 + (\omega_d)^2}\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\left[\frac{\omega_n\sqrt{1-\zeta^2}}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

$$\boxed{Y(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

Recall s-plane representation of 2nd order pole locations



Problem 2

Nise 4-25

(a) $\sum F = m\ddot{x} = -b\dot{x} - kx + f(t)$

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

$$X(s)(ms^2 + bs + k) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2+bs+k}$$

(b) Rewrite the char eqn: $s^2 + \frac{b}{m}s + \frac{k}{m} = 0$

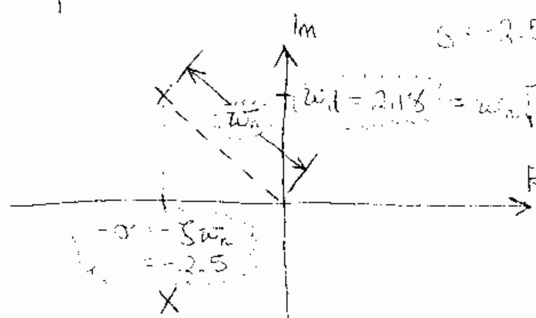
to match std form: $s^2 + 2\zeta\omega_n s + \omega_n^2$

Then, $\omega_n^2 = \frac{k}{m} \rightarrow \omega_n = \sqrt{\frac{k}{m}} = \mathbf{3.16} \left[\frac{rad}{s}\right]$

$2\zeta\omega_n = \frac{b}{m} \rightarrow \zeta = \frac{b}{2\omega_n m} = \frac{b}{2\sqrt{km}} = \mathbf{.754}$

... Recall how ζ & ω_n relate to the root locations on the s-plane:

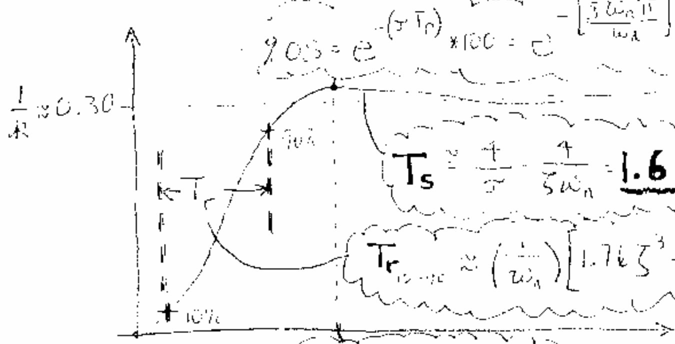
$s = -2.5 \pm 2.18j$



$$\%OS = e^{-(\sigma T_p)} \times 100 = e^{-\left[\frac{\zeta\omega_n \pi}{\omega_d}\right]} \times 100 = e^{-\left[\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right]} \times 100$$

only "times 100" to get percentage

$\%OS \cong$ **2.7%**

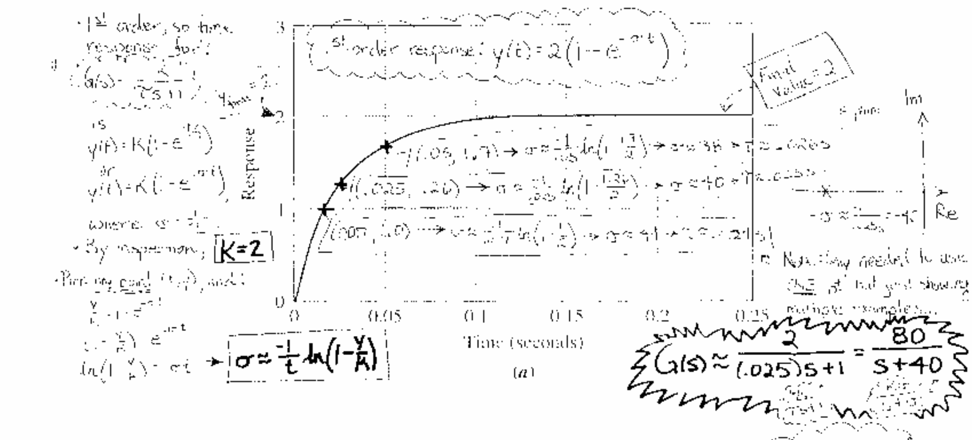


$T_s \cong \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} =$ **1.6 sec**

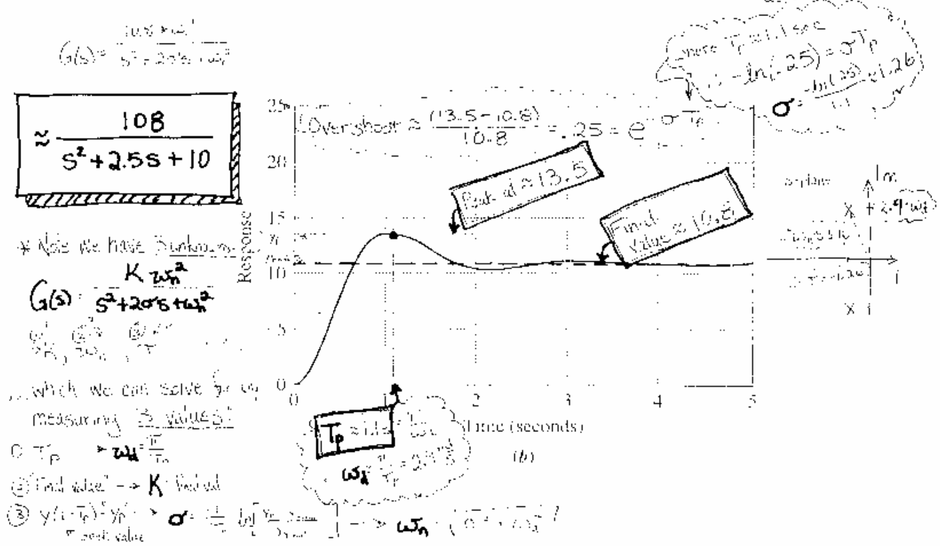
$T_{r_{10-90}} \approx \left(\frac{1}{\omega_n}\right)\left[1.76\zeta^3 - .417\zeta^2 + 1.039\zeta + 1\right] \approx$ **0.69 sec**

page 196, tables, graph, or eqn in footnote!!

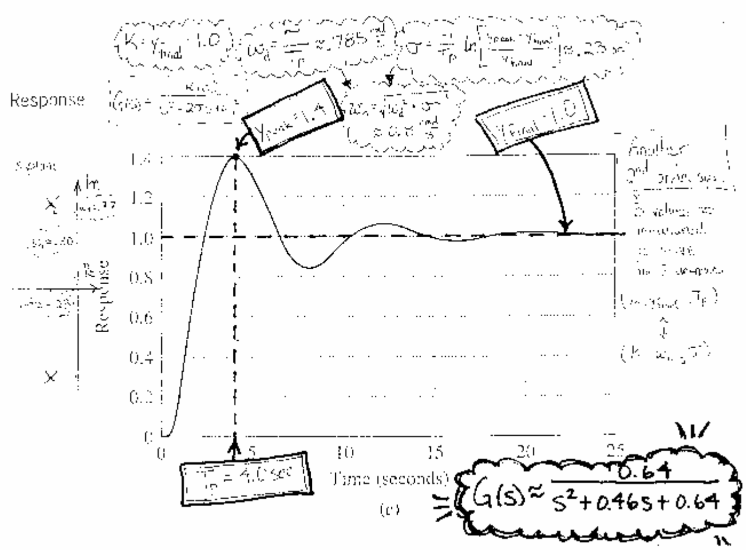
$T_p = \frac{\pi}{\omega_d} \approx$ **1.44 sec**


1st order response: y(t) = 2(1 − e^{−σt})
Final Value = 2
Response
Time (seconds)
K=2
σ ≈ −1/t ln(1 − y/K)
G(s) ≈ 2/((.025)s+1) = 80/(s+40)

(a)

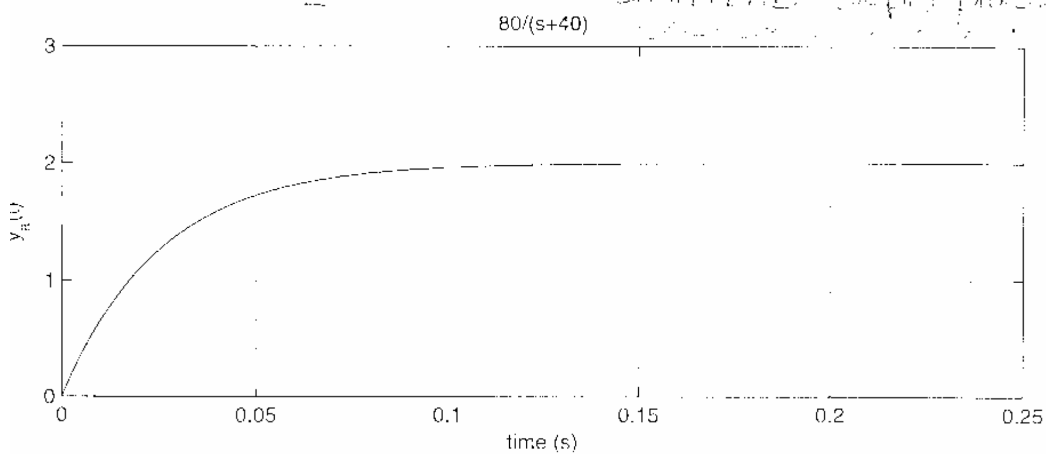

≈ 108/(s²+2.5s+10)
Overshoot ≈ (13.5 − 10.8)/10.8 = .25
Peak at ≈ 13.5
Final Value ≈ 10.8
Response
Time (seconds)
* Note we have 3 unknowns
...which we can solve for by measuring 3 values:

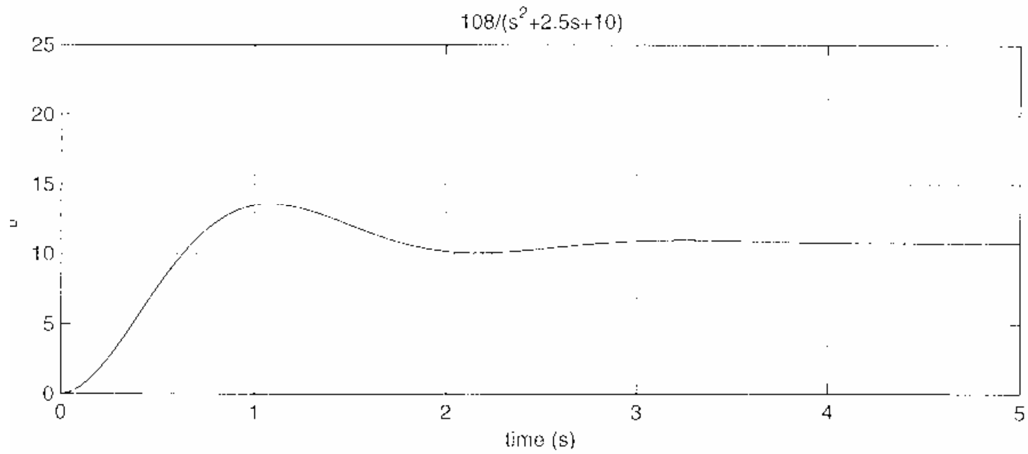
(b)

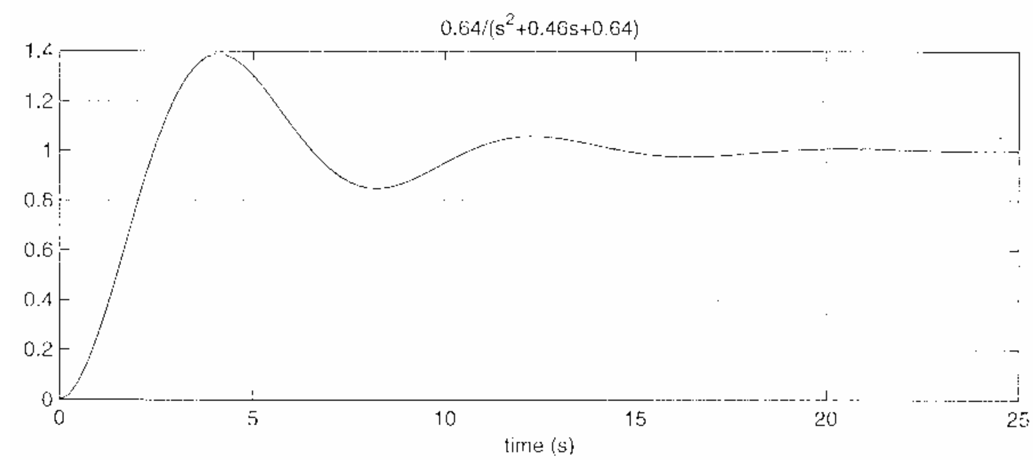

Response
y_peak = 1.4
y_final = 1.0
Time (seconds)
T_p = 4.0 sec
G(s) ≈ 0.64/(s²+0.46s+0.64)

(c)

Problem 1 MATLAB step() plots







Problem 4

(a) $G(s) = \dfrac{a}{(s+a)} \cdot \dfrac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)} = \dfrac{a}{(s+a)} \cdot \dfrac{c}{(s^2+bs+c)}$ Define $b = 2\zeta\omega_n$, $c = \omega_n^2$

if input is a unit step, multiply $\frac{1}{s} \cdot G(s)$:

$$\text{Output} = \frac{1}{s} \cdot \frac{a}{(s+a)} \cdot \frac{c}{(s^2+bs+c)} = \frac{K_1}{s} + \frac{K_2}{(s+a)} + \frac{K_3 s + K_4}{(s^2+bs+c)}$$

Cross-multiplying:

$$= \frac{(K_1+K_2+K_3+K_4)s^3 + (K_1(a+b)+K_2 b+K_3 a+K_4)s^2 + (K_1(ab+c)+K_2 c+K_4 a)s + K_1 ac}{s(s+a)(s^2+bs+c)}$$

$K_1 = 1$, by inspection (to make final value of step resp $\frac{ac}{ac}=1$)

To find the other 3 unknowns, solve 3 eqn's:

(eq 1) $K_2 + K_3 + K_4 = -1$
(eq 2) $bK_2 + aK_3 + K_4 = -(a+b)$
(eq 3) $cK_2 + aK_4 = -(ab+c)$

MATLAB can solve this system of eqn's symbolically using "solve( )"!

Solution is: $\boxed{K_2 = \dfrac{-c}{c-ab+a^2}}$ $\boxed{K_3 = \dfrac{a(b-a)}{c-ab+a^2}}$ $\boxed{K_4 = \dfrac{(b^2-c-ab)a}{c-ab+a^2}}$

Using $\omega_n = 10 \frac{rad}{s}$, $\zeta = .5$: $b = 2\zeta\omega_n = 10$, $c = \omega_n^2 = 100$

$K_2 = \dfrac{-100}{100-10a+a^2}$, $K_3 = \dfrac{10a - a^2}{100-10a+a^2}$, $K_4 = \dfrac{-10a^2}{100-10a+a^2}$

Using Laplace Table from prob #2 solutions,

(a) ⇨ $y(t) = 1 + K_2 e^{-at} + e^{-\frac{b}{2}t}\left[K_3\cos(\omega_d t) + \dfrac{K_4 - \frac{b}{2}K_3}{\sqrt{c}} \sin(\omega_d t)\right]$

(b) 1st order part has magnitude $M_1 = |K_2|$

2nd order part has magnitude $M_2 = \left|\sqrt{K_3^2 + \left(\dfrac{K_4 - \frac{b}{2}K_3}{\sqrt{c}}\right)^2}\right|$

(Since K_2, K_3, K_4 all share same denominator "$c-ab+a^2$" [or "$100-10a+a^2$"], mult thru by this...)

$$\left|\frac{M_1}{M_2}\right| = \frac{100}{(1.25a^4 - 15a^3 + 125a^2)^{1/2}}$$

To solve for $\left|\frac{M_1}{M_2}\right| = 1$, let's solve the equation $M_2^2 - M_1^2 = 0$

$[1.25a^4 - 15a^3 + 125a^2 - 10000 = 0]$ > Roots are: $a = \{-6.0\pm 4.07 \cdot 10.25, \ldots, 10\}$

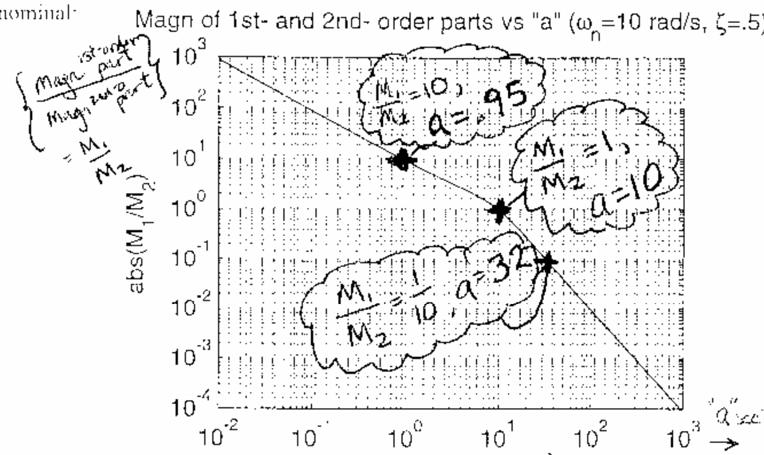
We need "a" to be a positive real number,

(b) ∴ $\boxed{a = 10}$ makes the effects of the 1st & 2nd order poles about the same.

Problem 4, continued... To answer parts (c) and (d), I wrote the MATLAB script shown below. I put in various values of "a" (also specifying wn and zeta), and the function calculates the "magnitudes" (Ca and Cwn) for the 1st-order and 2nd-order parts of the unit step response. Then I plotted the absolute value of the ratio of the two vs "a" on a log-log plot. I decided that one response "dominates" another when there is a factor of 10 difference between them, and those particular values of "a" are labeled on the plot at right, below. The plot would look somewhat different if zeta were changed! I thought it was interesting it was not simply a straight line (but instead "breaks" at a=10).

```
function [Ca,Cwn] = step3pole(a,wn,zeta);
% function [Ca,Cwn] = step3pole(a,wn,zeta);
%
% Take inputs a (zero location), wn and zet. Outputs the magnitudes of the 1st-order
% part of the time reponse (Ca) and of the 2nd-order part (Cwn). Also plots the two
% contributions (1st- and 2nd-order) and shows how they add to one another.

% Problem Set 3, problem 4 uses nominal:
% a=10; wn=10; zeta=.5;
m1=tf(wn^2,[1 2*zeta*wn wn^2])
m2=tf(a,[1 a]);
m3=tf([1],[1 0]);
[n,d]=tfdata(m1*m2*m3,'v');
[r,p]=residue(n,d);
pa=[]; pwn=[]; pz=[];
for n=1:4
   if isreal(p(n))
      if p(n)==0, pz=n;
      else, pa=n; end
   else
      pwn=[pwn n];
   end
end
r=r([pwn pa pz]);
p=p([pwn pa pz]);
n2=r(1)*[1 -p(2)]+r(2)*[1 -p(1)];
A=n2(1); B=n2(2);
Cwn=sqrt(A*A + ((B-zeta*wn*A)/wn)^2);
Ca=r(3);
d2=conv([1 -p(1)],[1 -p(2)]);
tmax=max(6/a,abs(6/real(p(1))));
[y1,t1]=impulse(tf(n2,d2),tmax);
[y2,t2]=impulse(tf(r(3),[1 -p(3)]),t1);
% figure(1); clf
p1=plot(t1,y1+y2+1,'k-');
set(p1,'LineWidth',2); hold on
plot(t1,y1+y2,'k-');
plot(t1,y1,'r-.');
plot(t2,y2,'b--'); grid on
legend('y_t_o_t(t)=1+y_1(t)+y_2(t)','y_1(t)+y_2(t)','y_1(t)','y_2(t)',4);
title(['a = ' num2str(a) ', \omega_n = ',...
      num2str(wn) ' (rad/s), \zeta = ' num2str(zeta) '.']);
xlabel('time (s)');
ylabel('unit step response');
```

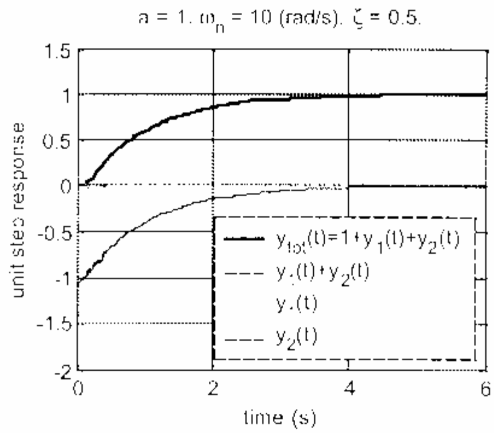


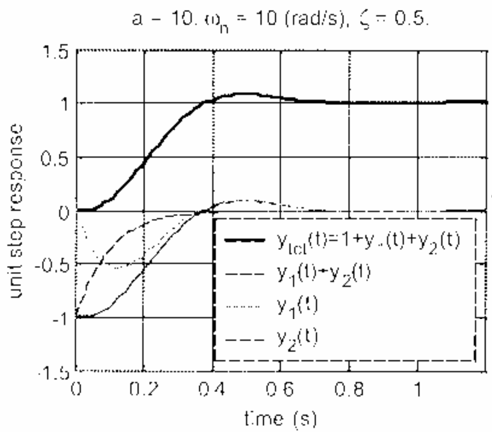
(c) When a = .95 sec^{-1}, the 1st-order part of the step response is 10x the magnitude of the 2nd-order part, so: $a < 1.0$ is a good range.

(d) When a=32 sec^{-1}, the 2nd-order part is 10x the magnitude of the 1st-order part. $a > 32$ is my choice.

(d) Here are my MATLAB plots, which were all generated using that code on the previous page. You can get that m-file from a link on the 2.14 Problems Sets page of our website.

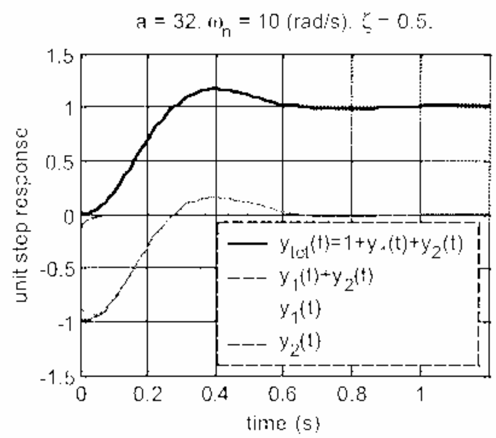
1^{st}-order dominates when "a" is "slow" compared with ? $_n$. plot shows case where $a = 1$





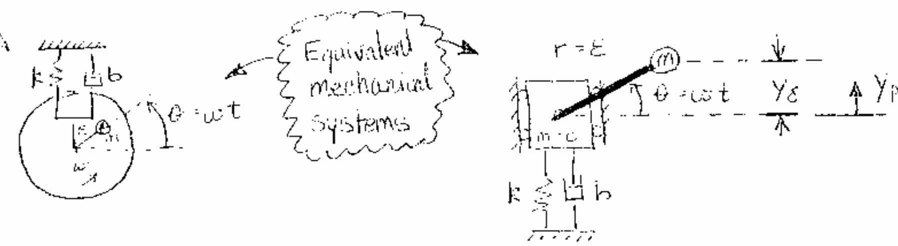
Both 1^{st}- and 2^{nd}- order poles contribute to the total response in approximately equal measure when "a" is near the same frequency as ? $_n$. $a = 10$

2^{nd}-order dominates when "a" is "fast" compared with ? $_n$. plot shows case where $a = 32$



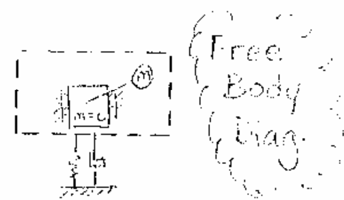
(e) In the intermediate case. I would use the same metric for settling time as for the 2^{nd}-order poles, alone. The time constant for the 1^{st}-order part (tau 1/a) is half that of the decay envelope for the second order system (given zeta .5). so it dies off much more rapidly. Also, the magnitude of the 2^{nd}-order sinusoid is somewhat smaller than that for a unit step response for the 2^{nd}-order system w/out the addition pole at s=-a, so using the estimate Ts 4*zeta*wn should be reasonable (and slightly conservative).

Problem 5

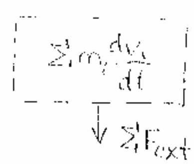


(a) First, to make clear we are only worrying about vertical motion, I've drawn an equivalent mechanical system (at right, above). The "cart" and link to the point-mass are massless. "y_P" refers to the vertical position of the "pivot" point, where the rod is mounted.

"y_ε" measures the difference between the vertical position of the mass (y_m) and of the pivot (and rest of the cart), y_P: $\boxed{y_\varepsilon = y_m - y_P}$



Next, I've drawn a F.B.D. that uses a control volume (arbitrary) that encloses the cart + mass.



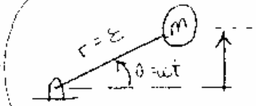
Using Newton's 2nd Law, the change in momentum of all mass enclosed in the C.V. [when summed up] equals the sum of all external forces applied to the C.V.

(If an astronaut pushes himself off the interior wall of a space station at rest, and there are no external forces on the space station (rocket thrusters, gravity force, etc), the astronaut may change his own momentum; $m_i \frac{dv_i}{dt}$ may be non-zero, that is — but the space station experiences an equal & opposite change in its momentum... no net total change in momentum for all mass in the C.V.)

[Markin 5]

$y_\delta = r\sin(\theta)$

$y_\delta = \varepsilon \sin(\omega t)$



$\frac{d}{dt}$: $\dot{y}_\delta = \omega \varepsilon \cos(\omega t)$

$\frac{d}{dt}$: $\ddot{y}_\delta = -\omega^2 \varepsilon \cos(\omega t)$

So: $m\frac{dv_m}{dt} = \sum F = -b\dot{y}_p - ky_p$

$$m(\ddot{y}_p + \ddot{y}_\delta) = -b\dot{y}_p - ky_p$$

$$m\ddot{y}_p + b\dot{y}_p + ky_p = -m\ddot{y}_\delta$$

$$\boxed{m\ddot{y}_p + b\dot{y}_p + ky_p = m\varepsilon\omega^2 \sin(\omega t)}$$

So it is equivalent to model a system where we use the position of the cart, y_p, as an output and use:

$$F_w(t) = m\varepsilon\omega^2 \sin(\omega t) \leftarrow$$

(b)

Then: $m\ddot{y}_p + b\dot{y}_p + ky_p = F_w(t)$

$$\boxed{G_p(s) = \frac{Y_p(s)}{F_w(s)} = \frac{1}{ms^2 + bs + k}}$$

← if "y" is taken to mean the place where b & k attach... "y_p"

But do we want to know $\frac{Y_p(s)}{F_w(s)}$ or $\frac{Y_m(s)}{F_w(s)}$? I'll accept either!

For $Y_m(s)$, note: $\frac{Y_m(s)}{F_w(s)} = \frac{Y_p(s)}{F_w(s)} + \frac{Y_\delta(s)}{F_w(s)}$

Using: $m = 2$ kg
$k = 200$ N/m
$b = 12$ N·s/m
$\varepsilon = .05$ m

note: $\frac{y_\delta(t)}{f_w(t)} = \frac{\varepsilon \sin(\omega t)}{m\omega^2 \varepsilon \sin(\omega t)} = \frac{1}{m\omega^2}$

$(j\omega) \to s$, so $[(j\omega)^2 = -\omega^2] \to s^2$, or $\{\omega^2 \to -s^2\}$

$$\left.\frac{Y_\delta(s)}{F_w(s)} = \frac{-1}{ms^2}\right\} \to \frac{Y_m(s)}{F_w(s)} = \frac{1}{ms^2 + bs + k} + \frac{-1}{ms^2}$$

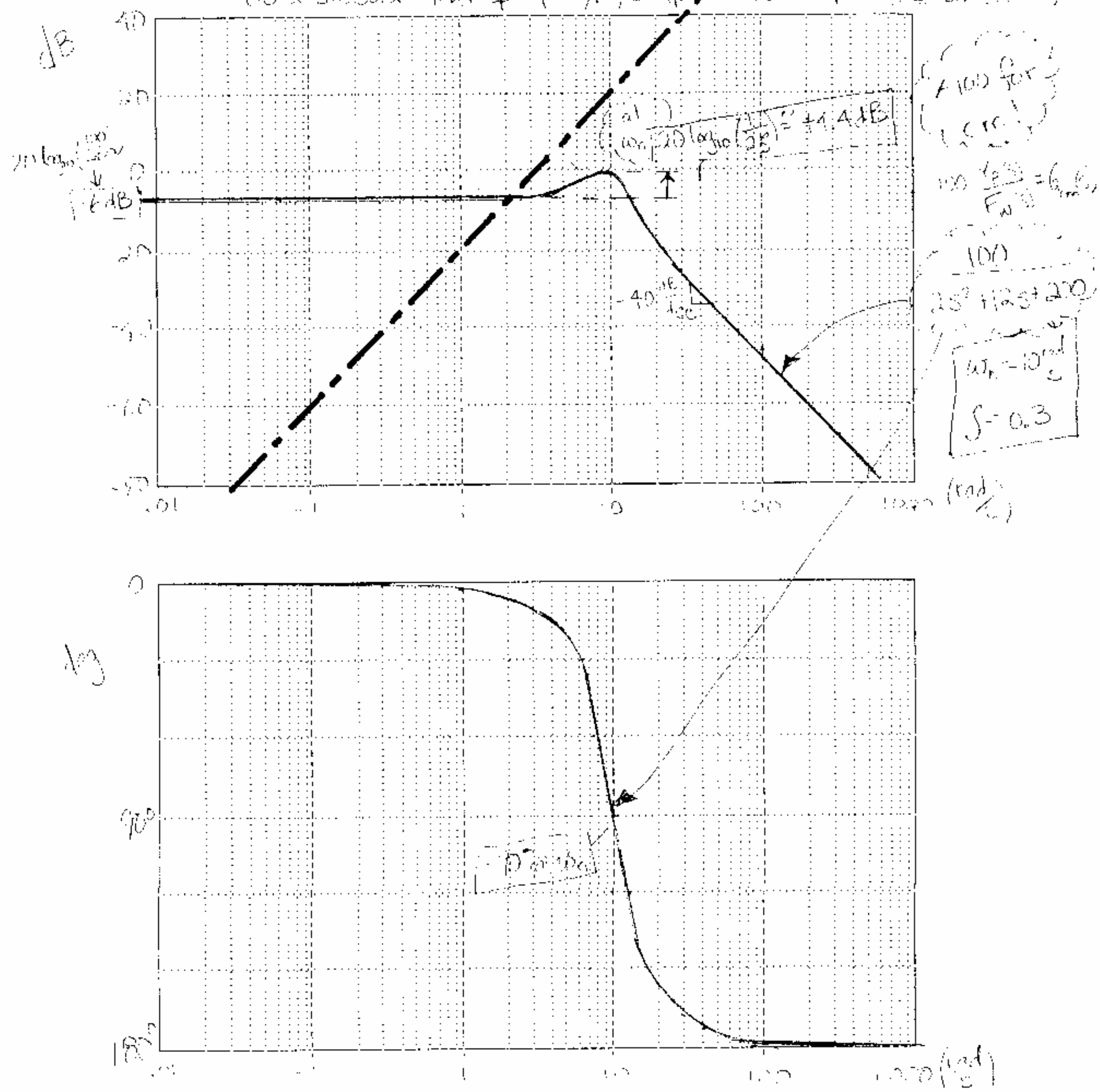
$$= \frac{ms^2 - (ms^2 + bs + k)}{ms^2 + bs + k}$$

$$= \frac{-(bs + k)}{ms^2(ms^2 + bs + k)}$$

[Problem 5]

(c) & (d) $F_w(\omega)$ goes as $[m\varepsilon\omega^2] = (0.1\,\omega^2)$

(is a sinusoid - freq resp implies/requires inputs and outputs ARE sinusoids!)



Problem 6

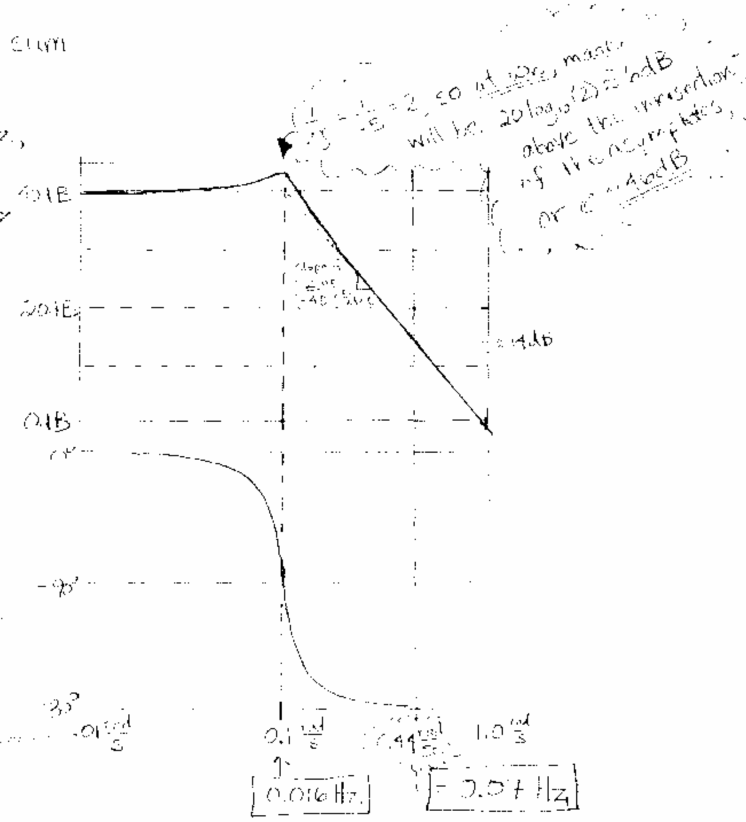
(a) Assume inputs are a sum of sinusoids.

(b) $\omega_n = 0.1 \frac{rad}{s} \approx 0.016$ Hz, [illegible] low freq mag. is $\frac{K}{\omega_n^2} = 100$, or 40 dB

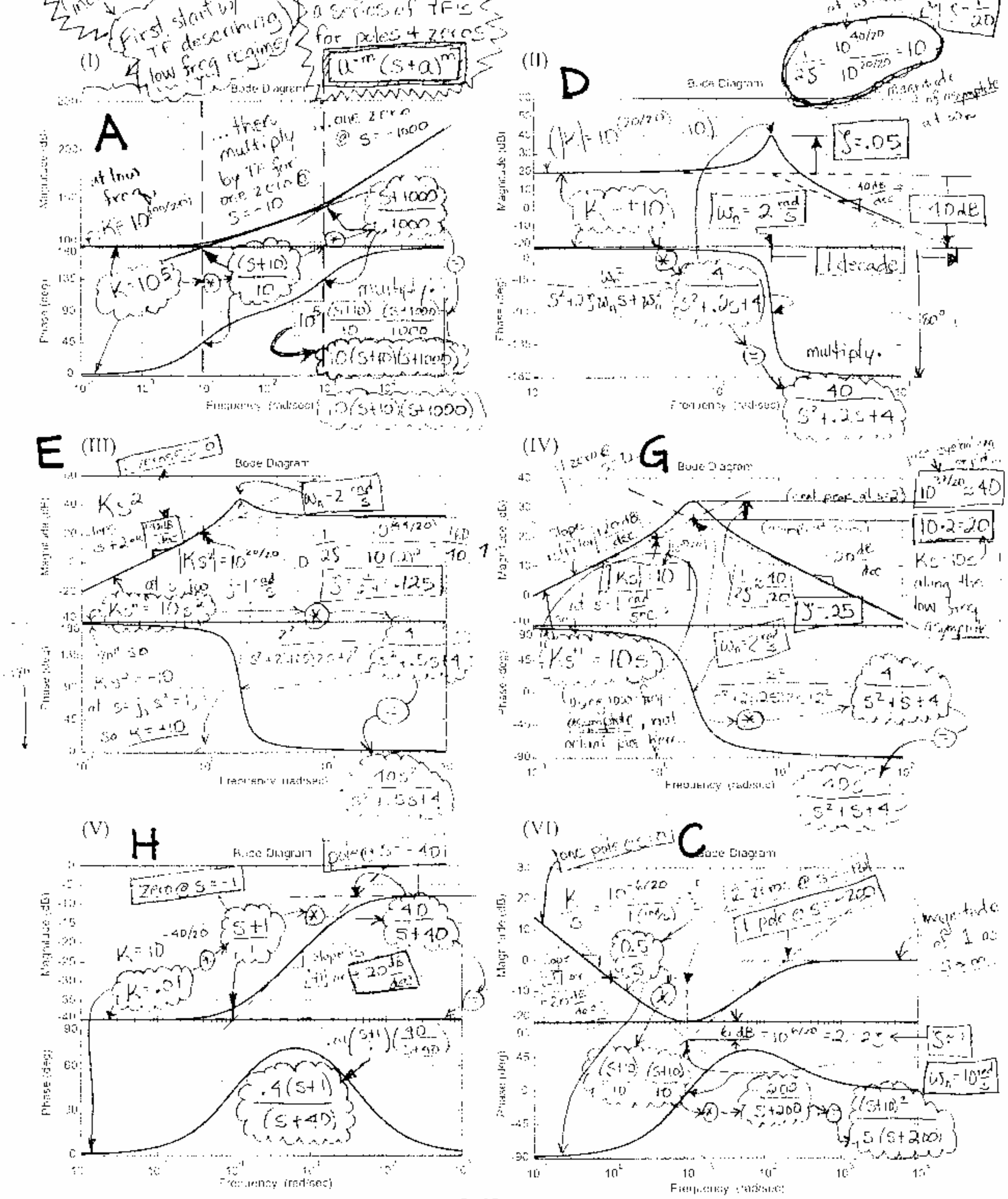
(c) "Worst case" looks like frequency near 0.07 Hz, or $\times 2\pi \rightarrow 0.44 \frac{rad}{s}$

(d) Would like to push ω_n even lower, so magnitude rolls off even more by .44 $\frac{rad}{sec}$. $\omega_n = \sqrt{\frac{K}{J}}$, so $K\downarrow$ or $J\uparrow$ will lower ω_n. High freq. asymp. goes as $\frac{K}{J\omega^2}$, so think $J\uparrow$ is better idea.

(K may give more $K\uparrow$ [illegible] stabilizers may also help [illegible])



Problem 7



Start with Ks^n includes any poles or zeros at s=0!!
First start w/ TF describing low freq regime
then put in higher freq stuff...
i.e., multiply by a series of TF's for poles + zeros
$a^{-m}(s+a)^m$
(I) A
Bode Diagram
at low freq, $K = 10^{100/20}$
$K = 10^5$
...then multiply by TF for one zero @ s = -10
$\frac{(s+10)}{10}$
one zero @ s = -1000
$\frac{s+1000}{1000}$
multiply
$10^5 \frac{(s+10)}{10} \frac{(s+1000)}{1000}$
$10(s+10)(s+1000)$
Frequency (rad/sec)
Magnitude (dB)
Phase (deg)
(II) D
$\zeta = .05$
$\omega_n = 2 \frac{rad}{s}$
-40dB
1 decade
$K = 10$
$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{4}{s^2+.2s+4}$
multiply.
$\frac{40}{s^2+.2s+4}$
(III) E
Ks^2
$\omega_n = 2 \frac{rad}{s}$
$\zeta = .125$
$\frac{4}{s^2+.5s+4}$
$\frac{10s^2}{s^2+.5s+4}$
(IV) G
$10^{32/20} \approx 40$
$10 \cdot 2 = 20$
$\zeta = .25$
$\omega_n = 2$
$Ks^1 = 10s$
$\frac{4}{s^2+s+4}$
$\frac{40s}{s^2+s+4}$
(V) H
pole @ s = -40
zero @ s = -1
$K = 10^{-40/20}$
$K = .01$
$\frac{s+1}{1}$
$\frac{40}{s+40}$
$\frac{.4(s+1)}{(s+40)}$
(VI) C
one pole @ s=0
2 zeros @ s = -10
1 pole @ s = -200
$\frac{0.5}{s}$
$\frac{(s+10)}{10} \frac{(s+10)}{10}$
$\frac{200}{s+200}$
$\frac{(s+10)^2}{.5s(s+200)}$
$\omega_n = 10 \frac{rad}{s}$