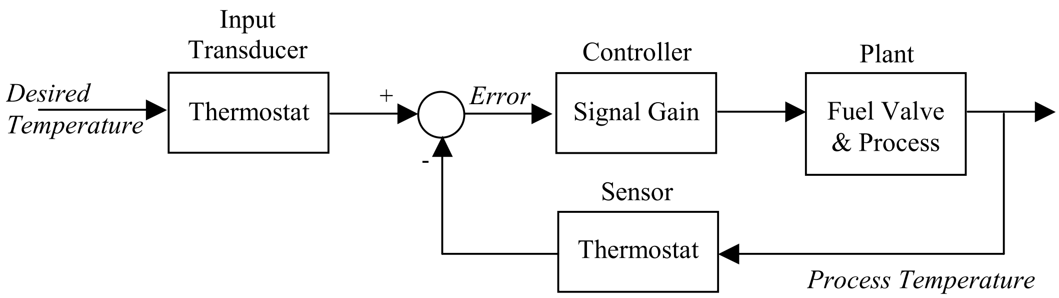
Department of Mechanical Engineering

2.14 ANALYSIS AND DESIGN OF FEEDBACK CONTROL SYSTEMS

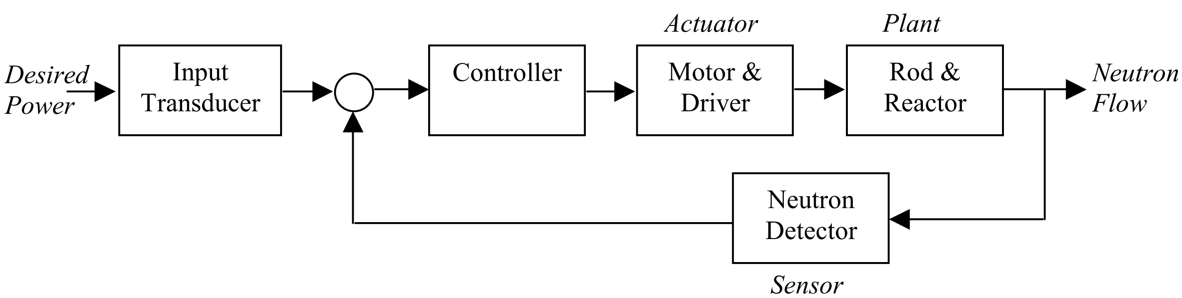
Fall Term 2003

Problem Set 1 Solution

Problem 1: (Nise, Ch. 1, Pr. 2)



Problem 2: (Nise, Ch. 1, Pr. 5)



Problem 3: Given the transfer function

$$G(S) = \frac{1}{(s+2)(s+6)}$$

(i) The step response using the Laplace transform

$$Y_{step}(s) = G(s)U_{step}(s) = \frac{1}{(s+2)(s+6)}\frac{1}{s}$$

Doing partial fraction expansion

$$Y_{step}(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

where,

$$A = sY_{step}(s)\Big|_{s=0} = \frac{1}{(s+2)(s+6)}\Bigg|_{s=0} = \frac{1}{12}$$

$$B = (s+2)Y_{step}(s)\Big|_{s=-2} = \frac{1}{(s)(s+6)}\Bigg|_{s=-2} = -\frac{1}{8}$$

$$C = (s+6)Y_{step}(s)\Big|_{s=-6} = \frac{1}{(s)(s+2)}\Bigg|_{s=-6} = \frac{1}{24}$$

therefore

$$Y_{step}(s) = \frac{1}{12s} - \frac{1}{8(s+2)} + \frac{1}{24(s+6)}$$

Taking the inverse Laplace transform (Nise, pp. 40, Table 2.1)

$$y_{step}(t) = L^{-1}\{Y_{step}(s)\} = \left[\frac{1}{12} - \frac{1}{8}e^{-2t} + \frac{1}{24}e^{-6t}\right]u_{step}(t)$$

(ii) The impulse response using Laplace transform.

$$Y_{imp}(s) = G(s)U_{imp}(s) = \frac{1}{(s+2)(s+6)}1$$

Doing partial fraction expansion

$$Y_{imp}(s) = \frac{A}{s+2} + \frac{B}{s+6}$$

where,

$$A = (s+2)Y_{imp}(s)\Big|_{s=-2} = \frac{1}{(s+6)}\Bigg|_{s=-2} = \frac{1}{4}$$

$$B = (s+6)Y_{imp}(s)\Big|_{s=-6} = \frac{1}{(s+2)}\Bigg|_{s=-6} = -\frac{1}{4}$$

therefore

$$Y_{imp}(s) = \frac{1}{4(s+2)} - \frac{1}{4(s+6)}$$

Taking the inverse Laplace transform (Nise, pp. 40, Table 2.1)

$$y_{imp}(t) = L^{-1}\{Y_{imp}(s)\} = \left[\frac{1}{4}e^{-2t} - \frac{1}{4}e^{-6t}\right]u_{step}(t)$$

Note that $y_{imp}(t)$ could be found by differentiating $y_{step}(t)$ with respect to t.

Problem 4: Given a car of mass *m*, driven by a force *F(t)* and with viscous resistance *B*.

(a) Differential equation relating the car velocity *v(t)* to the applied force *F(t)*. Doing and equilibrium of forces we obtain

$$F(t) - Bv(t) = ma(t) \text{ and } a(t) = \frac{dv(t)}{dt}$$

Rearranging terms

$$\frac{dv(t)}{dt} + \frac{B}{m}v(t) = \frac{F(t)}{m}$$

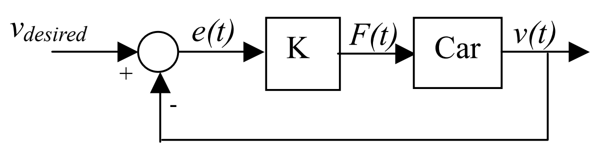
(b) The value of the coefficient *B* can be found by using the homogeneous solution $v_h(t)$ to the differential equation found in *(a)* and the initial condition v_o.

$$v_h(t) = e^{-\frac{B}{m}t} \text{ and } v(t) = v_o v_h(t) = v_o e^{-\frac{B}{m}t}$$

Given the initial condition $v_o=10m/s$, the mass of the car $m=1000kg$ and the speed $v(t)=0.2m/s$ at $t=8s$,

$$0.2 = 10e^{-\frac{B}{1000}(8)} \Rightarrow B = -\frac{1000}{8}\ln(\frac{0.2}{10}) = 489\, Ns/m$$

(c) Closed-loop system block diagram



where Car is described by the differential equation found in *(a)*

(d) To find the differential equation relating *v(t)* to $v_{desired}(t)$, modify the equation found in *(a)* by substituting $F(t)=K(v_{desired}(t)-v(t))$

$$\frac{dv(t)}{dt} + \frac{B}{m}v(t) = \frac{F(t)}{m} = \frac{K(v_{desired}(t) - v(t))}{m}$$

rearranging terms

$$\frac{dv(t)}{dt} + \frac{(B+K)}{m}v(t) = \frac{K}{m}v_{desired}(t)$$

(e) Transfer function is derived by taking the Laplace transform of the differential equation in *(d)*

$$L\left\{\frac{dv(t)}{dt}+\frac{(B+K)}{m}v(t)\right\}=L\left\{\frac{K}{m}v_{desired}(t)\right\}\Rightarrow sV(s)+\frac{B+K}{m}V(s)=\frac{K}{m}V_{desired}(s)$$

Therefore

$$H(s)=\frac{V(s)}{V_{desired}(s)}=\frac{K/m}{s+(B+K)/m}$$

(f) The closed-loop system time constant τ can be obtained from the closed-loop transfer function found in *(e)*

$$\tau=\frac{1}{(B+K)/m}=\frac{m}{B+K}$$

given $K=100Ns/m$, $m=1000kg$ and $B=489Ns/m$, then $\boldsymbol{\tau}$**=1.7s**.

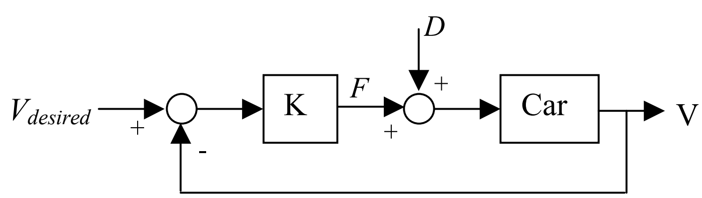
(g) The steady-state speed of the car on level ground is found using the *Final Value Theorem*

$$v_{ss}=\lim_{s\to 0}sV(s)=\lim_{s\to 0}sH(s)V_{desired}(s)=\lim_{s\to 0}s\frac{K/m}{s+(B+K)/m}\frac{V_{desired}}{s}=\frac{V_{desired}K}{B+K}$$

substituting values: $V_{desired}=60mph$, $K=100Ns/m$, and $B=489Ns/m$

$$v_{ss}=10.19mph$$

(h) To take into account the gravity effect on the system, let's consider it as a disturbance *D* applied between the controller and the car



where *Car* is described by the transfer function

$$H_{car}(s)=\frac{1/m}{s+B/m}$$

The transfer function relating *V(s)* to $V_{desired}(s)$ was found previously in *(e)*. The transfer function relating *V(s)* to *D(s)* is found from the block diagram above with $V_{desired}(s)=0$

$$V(s) = H_{car}(s)(-KV(s) + D(s)) \Rightarrow H_D(s) = \frac{V(s)}{D(s)} = \frac{1/m}{s + (B+K)/m}$$

The steady-state speed of the car is found by superposition using the result from *(g)* and the steady-state speed due to a step disturbance of magnitude $D=-mg\sin(15°)$

$$v_{ss} = v_{ss}\big|_{D=0} + v_{ss}\big|_{V_{desired}=0} = \frac{V_{desired}K}{B+K} + \frac{D}{B+K} = \frac{V_{desired}K}{B+K} - \frac{mg\sin(15^\circ)}{B+K}$$

Substituting values and converting to mph,

$v_{ss}=10.19-9.64=0.55mph$ (barely moving!)

<u>Problem 5</u>:

(i)

$$\frac{C(s)}{R(s)} = \frac{0.1K}{s^2 + 6s + 0.1K}$$

(ii)

$$\frac{C(s)}{D(s)} = \frac{0.1}{s^2 + 6s + 0.1K}$$

(iii)

$$R(s) = \frac{1}{s}, \quad c_{ss} = \lim_{s \to 0} sC(s) = 1$$

(iv)

$$D(s) = \frac{1}{s}, \quad c_{ss} = \lim_{s \to 0} sC(s) = \frac{1}{K}$$